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ALCUBIERRE'S DRIVE: A STUDY USING ADM FORMALISM AND ENERGY CONDITIONS

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Content

Introduction

- “Nothing can travel faster than the speed of light" is a the postulate of General relativity.
- The postulate is constrained to inertial frames which exists locally as stated by General Relativity.
- Miguel Alcubierre in 1994 proposed the idea of Faster than Light (FTL) travel within the framework of General Relativity.

- Two observers start there motion at time t=0.
- The relative speed of separation of two comoving observers is greater than the speed of light. [1]
- This explains how a travel greater than speed of light is possible.

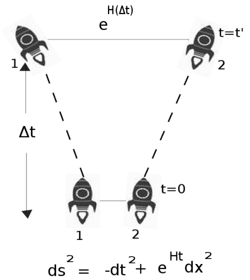

Figure: Comoving observers : Speed of separation is greater than c

Alcubierre's Drive

- A local distortion of spacetime can make a spaceship move faster than the speed of light.
- An expansion of spacetime behind the vehicle and contraction in front of it pushes the spaceship to move in forward direction.
- An inverse process can bring back the spaceship to the starting point.
- This way arbitrary speed and a lesser time to complete a round trip journey is obtained.

Alcubierre's drive

- The motion of a spaceship is described by the distortion of spacetime.
- The metric of spacetime is given by,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$= -(\alpha^2 - \beta_i \beta^i) dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j$$

- We need a 3+1 formalism to explain this metric.

ADM formalism

- In this formalism the spacetime is defined by a foliation of spacelike hypersurfaces of constant time t.
- Each surface is a “leaf of foliation”.
- There is a unit timelike vector which is normal (n^{μ}). This vector may not be normal to the hypersurface of a different set of observers.

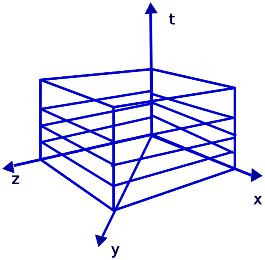

Figure: ADM splitting

ADM formalism

- For each surface, we can define a timelike vector t^μ.
- The tangential component of the vector is called the shift vector $N^\alpha = t_{||}$. The normal component of t^μ define the "distance between the hypersurfaces" called the lapse function, $N = t_\perp$.
- Thus, the timelike vector t^μ can be written as,

$$t^\mu = Nn^\mu + N^\mu$$

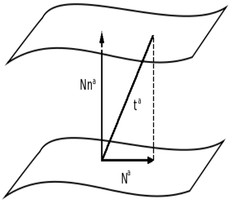

Figure: Leaf of foliation[2]

Figure: *

Source: https://inspirehep.net/record/1281391/files/foliation2.png

ADM splitting

- The components of the metric of the splitting is given by,

$$\begin{aligned} g_{00} &= g_{\mu\nu} t^{\mu} t^{\nu} \\ &= -N^2 + N^{\mu} N_{\mu} \end{aligned}$$

- The other components of the metric is obtained as,

$$g_{\mu\nu} t^{\mu} N^{\nu} = N^{\mu} N_{\mu} = N^{\alpha} N_{\alpha}$$

- Thus the complete metric can be written as,

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N^{\alpha} N_{\alpha} & \mathbf{N} \\ \mathbf{N}^T & g_{ab} \end{pmatrix}$$

where $a, b \in \{1, 2, 3\}$ and $\mathbf{N} \equiv N^{\alpha}$

- The line element is given by,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$= (-N(t)^2 + N^\alpha N_\alpha) dt^2 + 2N_\alpha dx^a dt + g_{ab} dx^a dx^b$$

- Stress energy tensor $T^{\mu\nu}$ describe the energy density, pressure and stress of a system.
- $T^{\mu\nu}$ can be defined as “a (2, 0) tensor with units of energy per volume, which is conserved." [3]
- It appears on the right hand side of Einstein’s equation representing how matter is related with spacetime.

Energy Conditions

- Energy conditions are coordinate invariant restrictions on the energy momentum tensor.

- The **Weak Energy Condition** (WEC) states that $T_{\mu\nu}t^{\mu}t^{\nu} \geq 0$ for all timelike vector t^{μ} or equivalently $\rho \geq 0$ and $\rho + p \geq 0$.

- The **Null Energy Condition** (NEC) states that $T_{\mu\nu}l^{\mu}l^{\nu} \geq 0$ for all null vectors l^{μ}, or equivalently that $\rho + p \geq 0$.

- The **Dominant Energy Condition** (DEC) contains the WEC, as well as the additional requirement that $T^{\mu\nu}t_{\mu}$ is a nonspacelike vector (namely, that $T_{\mu\nu}T^{\nu}_{\lambda}t^{\mu}t^{\lambda} \leq 0$).

- The **Null Dominant Energy Condition** (NDEC) is the DEC condition for null vectors only: for any null vector l^μ, $T_{\mu\nu} l^\mu l^\nu \geq 0$ and $T^{\mu\nu} l_\mu$ is a nonspacelike vector.

- The **Strong Energy Condition** (SEC) states that $T_{\mu\nu} t^\mu t^\nu \geq \frac{1}{2} T^\lambda_\lambda t^\sigma t_\sigma$ for all timelike vectors t^μ, or or equivalently that $\rho + p \geq 0$ and $\rho + 3p \geq 0$.

Alcubierre's drive

- Alcubierre metric of spacetime is given by,

$$
\begin{aligned}
ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\
&= -(\alpha^2 - \beta_i \beta^i) dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j
\end{aligned}
$$

- The motion of the spaceship is assumed to be in x axis of the Cartesian coordinate system.

- The properties of the metric are ($G = c = 1$),

$$\alpha = 1$$

$$\beta^x = v_s(t) f(r_s(t))$$

$$\beta^y = \beta^z = 0$$

$$\gamma_{ij} = \delta_{ij}$$

where

$$v_s(t) = \frac{dx_s(t)}{dt}, \qquad r_s(t) = [(x - x_s(t))^2 + y^2 + z^2]^{1/2}$$

and where f is the function:

$$f(r_s) = \frac{tanh(\sigma(r_s + R)) - tanh(\sigma(r_s - R))}{2tanh(\sigma R)}$$

where $R > 0$ and $\sigma > 0$. For large values of σ the function becomes a "top hat" function:

$$\lim_{\sigma \to \infty} f(\sigma) = \begin{cases} 1 & \text{for } r_s \in [-R, R] \\ o & \text{otherwise} \end{cases}$$

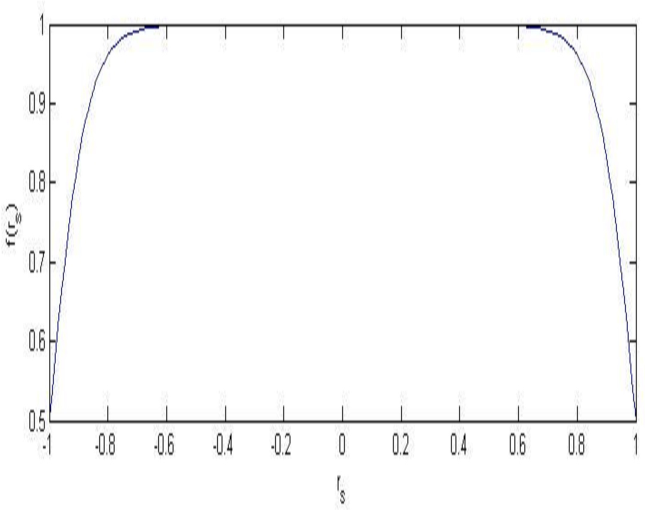

Figure: A plot between $f(r_s)$ and r_s

- The curvature of spacetime is defined by the extrinsic curvature tensor K_{ij}, which is given by,

$$K_{ij} = \frac{1}{2\alpha}(D_i\beta_j + D_j\beta_i - \frac{\partial\gamma_{ij}}{\partial t})$$

- Substituting for α and γ_{ij} we get,

$$K_{ij} = \frac{1}{2}(\partial_i\beta_j + \partial_j\beta_i)$$

Raychaudhuri Equation

- Newtonian fluid mechanics deals with the motion of fluid elements. It is similar to congruences in general relativity.
- Geodesic congruence is a bunch of non intersecting geodesics.
- The study of fluid elements is a tangential path in achieving the parameters required in the case of congruences.

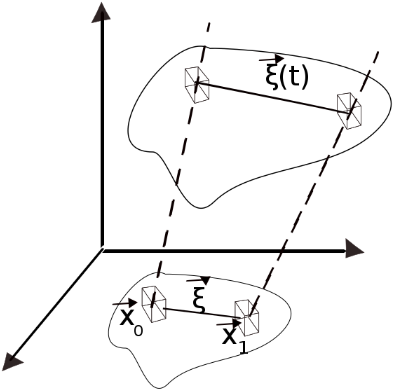

Figure: Fluid elements and fluid

- The displacement vector $\overrightarrow{\xi}$ is given by

$$\overrightarrow{\xi} = \overrightarrow{x_1} - \overrightarrow{x_0}$$

- The relative velocity between the fluid elements is given by,

$$\frac{d\overrightarrow{\xi}}{dt} = \overrightarrow{V}(\overrightarrow{x_1}) - \overrightarrow{V}(\overrightarrow{x_0})$$

- In components the relative velocity can be written as,

$$\frac{d\xi^{\alpha}}{dt} = V^{\alpha}(\overrightarrow{x_0} + \overrightarrow{\xi}) - V^{\alpha}(\overrightarrow{x_0})$$

- On Taylor expansion,

$$\frac{d\xi^\alpha}{dt} = V^\alpha{}_{,\beta}\xi^\beta + O(\xi^2)$$

- The rate of change of displacement vector is directly proportional to itself with a factor of gradient of the field evaluated at the point x_0.
- The equation becomes,

$$\frac{d\xi^\alpha}{dt} = B^\alpha{}_\beta \xi^\beta$$

where, $B_{\alpha\beta} = V_{\alpha,\beta}(\overrightarrow{x_0})$ represents the gradient of the velocity vector field evaluated at the reference fluid element.

- $B_{\alpha\beta}$ is a (3X3) matrix without symmetries which breakdowns into a diagonal part, symmetric trace free part and an anti symmetric part.

$$B_{\alpha\beta} = \frac{1}{3}\delta_{\alpha\beta}\Theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

where,

$\Theta = \delta^{\alpha\beta} B_{\alpha\beta} \implies$ the expansion scalar.

$\sigma_{\alpha\beta} = B_{(\alpha\beta)} - \frac{1}{3}\delta_{\alpha\beta}(\theta) \implies$ the shear tensor.

$\omega_{\alpha\beta} = B_{[\alpha\beta]} \implies$ the rotation tensor.

- Expansion is given by the trace of the velocity gradient.

$$\Theta = \delta^{\alpha\beta} V_{\alpha,_\beta} = \vec{\nabla}.\vec{V}$$

- From continuity equation, the expansion parameter can be derived as,

$$\Theta = \frac{1}{\delta V}\frac{d}{dt}\delta V$$

- The scaling of the fluid is described by the expansion parameter.
- $B_{\alpha\beta}$ is a linear superposition of these separate actions coming from each piece.

- Frobenius' theorem which states that, the congruences of timelike geodesics is hypersurface orthogonal iff the rotation tensor $\omega_{\alpha\beta}$ vanishes.

$$\omega_{\alpha\beta} = 0$$

- Raychaudhuri equation is the evolution equation of the irreducible quantities of the tensor field $B_{\alpha\beta}$.

$$\frac{DB_{\alpha\beta}}{d\tau} = B_{\alpha\beta;\mu} U^{\mu} \qquad = U_{\alpha;\beta\mu} U^{\mu}$$

- On further simplification,

$$\frac{DB_{\alpha\beta}}{d\tau} = -B_{\alpha\mu} B_{\mu\beta} - R_{\alpha\mu\beta\nu} U^{\mu} U^{\nu}$$

- Taking the trace of the equation leads to,

$$\frac{dB}{d\tau} = -B^{\alpha\mu}B_{\mu\alpha} - R_{\mu\nu}U^{\mu}U^{\nu}$$

- This reduces to Raychaudhuri equation,

$$\frac{d\Theta}{d\tau} = -\frac{1}{3}\Theta^2 - \sigma^{\alpha\mu}\sigma_{\mu\alpha} + \omega^{\alpha\mu}\omega_{\mu\alpha} - R_{\mu\nu}U^{\mu}U^{\nu}$$

- In Alcubeierre's drive the expansion of the volume elements associated with the observers,

$$\theta = -\alpha \; TrK$$

$$\vartheta = -\alpha \operatorname{Tr}(K)$$

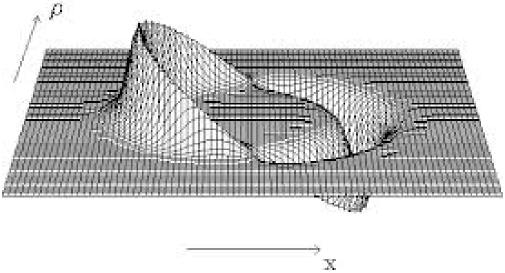

Figure: Expansion of the normal volume elements [1]

Figure: *

Source:http://www.npl.washington.edu/AV/warp.gif

A graph of θ and $\rho = (y^2 + z^2)^{1/2}$ for $\sigma = 8$ and $R = r_s = 1$ is shown.

Casimir Effect

- Casimir Effect is a small attractive force acting between two closed parallel, uncharged plates.
- It is a physical force arising from a quantized field and due to quantum vacuum fluctuations of the electromagnetic field.[4][5]
- The vacuum is defined as a space filled with virtual particles, which are continuous state of fluctuations.
- Virtual particles follow Heisenberg uncertainty principle and can exist only in the time,

$$\Delta E \Delta t \approx \hbar$$

- Two parallel, uncharged metallic plates are placed with few micrometers apart in vacuum.
- There is no external electromagnetic field.
- Using the theory of quantum electrodynamics, the plates do affect virtual photons in the space and generate a net force on the plates.

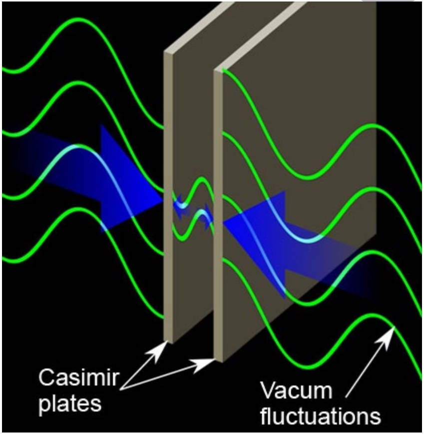

Figure: Casimir Effect [6]

- The attractive force between the two plates,

$$F = -\frac{\partial \Delta E}{\partial L} = -\frac{\pi \hbar c}{12L^2}.$$

- In three dimensions,

$$F_{cas} = -\frac{\partial \Delta E}{\partial L} = -\frac{\pi \hbar c}{240L^4}A \quad and \quad E_{cas} = -\frac{\pi \hbar c}{720L^3}A$$

  where A is the area of the plates.

- Casimir effect can produce a locally mass-negative region of spacetime which can be used to define wormholes and faster than light travel.
- Casimir effect arises in space-times with non-trivial topology.

Alcubierre's Drive

Results

- The metric describing the distortion of the spacetime, proposed by Alcubierre, violates the energy conditions (weak, dominant and strong).
- This implies the existence of exotic matter for Faster than Light (FTL).
- The presence of dark matter as explained by Quantum field theory allows the possibility of hyper-fast travel as suggested by Alcubeirre.

Conclusions

- A detailed study on Alcubierre's paper and its possibilities were conducted.
- A deep understanding of Energy conditions was done with the project.
- Basics of ADM splitting was studied.
- Raychaudhuri equation and Casimir effect was studied in depth.

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Thank you!