DYNAMICS OF
NON-INTEGRABLE SYSTEM

Project Report

submitted in partial fulfillment of the requirements for the degree of

MASTER OF PHYSICS

IN

PHYSICS

by

MERIN JOSEPH

Reg. No:15401215PH13

Under the Guidance of

Dr.DEEPAK VAID

DEPARTMENT OF PHYSICS

NATIONAL INSTITUTE OF TECHNOLOGY

KARNATAKA, SURATHKAL, MANGALORE-575 025

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DECLARATION

I hereby declare that the report of the P.G. Project Work entitled **" DYNAMICS OF NON-INTEGRABLE SYSTEM"** which is submitted to the National Institute of Technology Karnataka, Surathkal, in partial fulfillment of the requirements for the award of the Degree of Master of Science in the Department of Physics, is a bonafide report of the work carried out by me. The material contained in this report has not been submitted to any University or Institution for the award of any Degree.

Place:

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Department of Physics

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CERTIFICATE

This is to certify that the project entitled, **"DYNAMICS OF NON-INTEGRABLE SYSTEM"** is an authenticated record of work carried out by **MERIN JOSEPH**, Reg. No:15401215PH13 in partial fulfillment of the requirement for the award of the Degree of Masters of Science in Physics which is submitted to Department of Physics, National Institute of Technology Karnataka, during the period 2015-2016.

Dr.DEEPAK VAID

Project Advisor

Chairman-DPGC

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I would like to record my heartfelt thanks to Safir T K, Ahmed Rizwan C L and Rajani K V, research scholars, for their support and guidance in all times of need. Finally, I wish to express my deep thanks to my friends, for their timely assistance and cheerful co-operation. Above all, I thank my parents and the God Almighty for their blessings which helped in the successful completion of the project.

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ABSTRACT

The research field of non linear dynamics developed within last three decades. The deterministic nature of systems is termed as chaos. The systems which shows unpredictability of their future are called non integrable systems.

This is study on the Hamiltonian chaos and non integrable systems taking the case of spinning tops. The integrability of the top and ways to make it non-integrable is studied. The chaotic property of the system is studied using phase space plots, Lyapunov exponents and power spectrum.

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Chapter 1

Scope and Objective

Chaos is everywhere, from nature's most intimate considerations to art of any kind. The study of nonlinear dynamics or chaos theory has emerged in the last three decades or so as an important interdisciplinary area of research encompassing a wide range of fields like: fluids, bio-medical sciences, finance, turbulence, astronomy, material sciences, etc.
The study of rigid bodies is an inevitable part in physics. Most of the planetary system can be approximated to rigid bodies. One of such rigid body is spinning top. In this work I studied the dynamics of spinning top and its behavior when it is made non-integrable.

Objectives of my study are:

- To study the basics of Chaotic theory and dynamics of non integrable system using Hamiltonian formalism.
- To study numerically and analytically on a integrable system: Lagrange's top
- To study the non-integrability of tops numerically.

Chapter 2

Introduction

The world around us seems to be irregular and random. Chaos is the complicated temporal behaviour of simple systems. Chaos is a type of temporal evolution or dynamics. The study of dynamical system began in mid 1600s with Sir Isaac Newton. He discovered the laws of motion an gravitational laws and combined them to explain Kepler's planetary laws. In short Newton solved two body problem and those who preceded him tried to extend Newton's analytical methods to solve three body problem.

This problem was solved by Poincare in 1800s. He introduced a new point of view that emphasized qualitative rather than quantitative questions. For example, instead of asking for the exact positions of the planets at all times, he asked "Is the solar system stable forever, or will some planets eventually fly off to infinity?" (Strogatz, 2001). He was the first person to introduce the concept of chaos where in which a deterministic system exhibits aperiodic behavior that depends sensitively on the initial conditions, thereby rendering long-term prediction impossible. At the turn of the 20th century, Aleksandr Lyapunov worked on stability theory emphasizing quantitative evaluation of the divergence rate between solutions with different initial conditions. From the postwar period to the 1960s Andrei N. Kolmogorov and his students worked on the stability of Hamiltonian systems and developed the KAM theorem.

In 1954 Kolmogorov showed that a quasi-periodic regular motion can persist in an integrable system even when a slight perturbation is introduce to the system. This is known as the KAM (Kolmogorov- Arnold-Moser) theorem which indicates the limits to integrability(**?**). The theory also describes a progressive transition towards chaos. For an integrable system all trajectories are periodic or quasi-periodic. If a significant perturbation is introduced the probability of a quasi-periodic behaviour decreases and the trajectories become chaotic.

Edward Lorenz is the official discoverer of chaos theory. He first observed the phenomenon as early as 1961 and he discovered by chance what would be called later the chaos theory, in 1963, while making calculations with uncontrolled approximations aiming at predicting the weather (**?**). Lorenz developed the pictorial representation of his analysis and this figure is called Lorenz attractor. The strange attractor is a representation of a chaotic system in a specific phase space. More development to the subject came with the introduction of period doubling and logistic mapping. Mitchell Jay Feigenbaum proposed the scenario called period doubling to describe the transition between a regular dynamics and chaos. His proposal was based on the logistic map introduced by the biologist Robert M. May in 1976.

While the development of chaos theory went on there was work going on to quantize the non integrable systems. Einstein's paper on quantization of mechanical system brought into light the limitations of quantum theory when applied to a mechanical system which is non integrable or chaotic system. It was Martin Gutzwiller who solved this limitation and founded quantum chaos.

This project work on the non integrable systems in the light of Hamiltonian mechanics.

Chapter 3

Methodology

3.1 Symmetric top:The Integrable System

Symmetric top is an integrable system with two degrees of system.A rigid body which is symmetric which has a fixed point,called apex, on the symmetry axis is called symmetric top.The top is assumed to be spinning with its apex touching a plane sufficiently rough to prevent slipping, so that, in effect, the motion takes place with respect to a fixed point O coinciding with the point of contact.

A time-independent Hamiltonian system is said to be integrable if it has N independent global constants of the motion $f_i(p,q)$, i = 1,2,..., N (where $f_1 = H$) and

$$[f_i, f_j] = 0. \tag{3.1}$$

for all i and j. The requirement that an integrable system has N independent constants of the motion implies that the trajectory of the system in the phase space is restricted to lie on the N-dimensional surface

$$f_i(p,q) = k_i. \tag{3.2}$$

where i = 1,2,...,N and N and k_i are constants. If the system is integrable, the n first integrals can be used to define the n-dimensional manifolds $\Gamma_c = \{(p,q)|f_i(p,q) = k_i, 1 \leq j \leq n\}$ which is an N-dimensional torus. The n-torus is a naturally periodic object and can be considered as the direct product of n independent 2π periodicities. There can be N topologically independent closed paths γ_i on the torus where none of the γ_i can be deformed continuously into each other or shrunk to zero.

We need special techniques to study the integrability of a rigid body system.For this we use a moving frame, a relative frame whose coordinates are attached to the rigid body, and an absolute frame both having their origin at O. If R(t) is the rotation of the rigid body (moving frame) with respect to the absolute frame at t then the motion of the body is given by,

$$q(t) = R(t)Q. \tag{3.3}$$

For rigid bodies the position of the body with respect to the fixed system of co-ordinates is determined from the position of moving system. A rigid body has six degrees of freedom, the three independent angles and three components of position vector. The dynamics of rigid body are given by the equations of the form,

$$\gamma = \Gamma + \Omega \times r. \tag{3.4}$$

where γ is the dynamic quantity corresponding to moving frame *Gamma* corresponds to the rigid frame and r is such that $\dot{r} = \gamma$ and Ω is the angular velocity.

The total energy of the system is given by,

$$E = \frac{1}{2}M.\Omega + V. \tag{3.5}$$

Here V is due to the gravitational field and it remains constant. $M.\Omega$ is the momentum with respect to the vertical, so they are also constant. So the system is Hamiltonian and has one trivial first integral ,the Hamiltonian itself. For a system to be integrable we need another first integral that is commuting with H. There is a list of systems that satisfy the condition and are integrable.

1. *Euler-Poinsot top* In this case the fixed point is the center of mass (O=G) or L=0. The kinetic energy(K) is constant. The K and H are commuting.

2. *Lagrange top* This is the case of symmetric top where the inertia matrix is:

$$I = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_z \end{bmatrix}$$

   The line joining the center of mass and origin is the axis of rotation. The momentum K=M.L where the, M is the moment of motion, with respect to the axis is a first integral and it do commutate with the Hamiltonian. This case is more commonly known as symmetric top.

3. *Kowalevski case* There exist a moving frame in which the inertia matrix is

$$I = \begin{bmatrix} 2m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix}$$

   This is also integrable case.

If we consider a system in which the 3D bodies move such that there is always a point O which is fixed. The Euler theorem states that the general displacement of a rigid body with one fixed point is a rotation about some axis.

3.2 Non-Integrable Rigid Body Systems

For a non integrable system the second integral (I) of motion does not exist.That is only first integral that is the Hamiltonian is present. This opens up the path to chaos. It implies that trajectories will not be confined to a particular tori. The three cases we discussed in the last section are the integrable systems available in rigid body.

The system can be made non- integrable by various techniques. If we consider the Lagrange's top we can make it non integrable by two ways.

1. By breaking its symmetry.

   If the symmetry of Lagrange's top is broken asymmetric pendulum is formed where $I_x \neq I_y \neq I_z$. This change turns the system non-integrable and thus the system starts showing chaotic behavior.

2. By adding external stimulus.

   By giving an external multiple periodic perturbation to the system we can turn the integrable system to non integrable.

   As these systems turn non integrable then it can be treated as any other chaotic system and can be done analysis using chaotic tools.

3.3 Dynamics of rigid bodies

The dynamics of a rigid body is studied using the Euler equations in euler angles. The euler angles can describe the system. These are found by solving the equations for conservation of angular momentum, that is the Euler equations. There are 3 degrees of freedom needed to

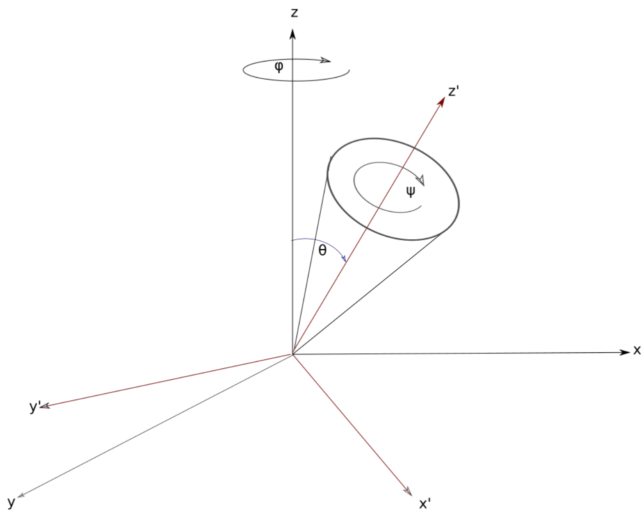


Figure 3.1: Schematic representation of a rigid body with euler angles

describe the position of the spinning top. They are θ, ϕ, ψ. θ is the angle between the z -axis of the space frame and the z'-axis of the body frame; ψ , which is the rotation angle of the z'-axis of the body frame around the z-axis of the space frame; and ϕ, which is the rotation angle of the spinning top around its own z'-axis (that of the body frame).

The angular velocity is given by $\omega_x, \omega_y, \omega_z$ as,

$$\begin{aligned} \omega_x &= \dot{\phi} \sin\theta \sin\phi + \dot{\theta}\cos\psi. \\ \omega_x &= \dot{\phi} \sin\theta \cos\phi - \dot{\theta}\sin\psi. \\ \omega_x &= \dot{\phi} \cos\theta + \dot{\psi}. \end{aligned} \tag{3.6}$$

We consider a symmetric body, appropriate for a top, for which the moments of inertia $I_x = I_y = I_0$ and $I_z = I$. The angular momentum is then

$$\begin{aligned} H_x &= I_0\omega_x. \\ H_y &= I_0\omega_y. \\ H_z &= I\omega_z/ \end{aligned} \tag{3.7}$$

The rotation of the body is studied using the Euler equations. The Euler equation is obtained from the below equation.

$$\dot{\mathbf{H}} = \frac{d}{dt}\mathbf{H} + \Omega \times \mathbf{H}. \tag{3.8}$$

where $\mathbf{H}$ is the applied torque and Ω is the total angular velocity of the rotating body. The Euler equation for the rigid body is given by,

$$\begin{aligned} M_x &= I_0(\ddot{\theta} - \dot{\phi}\sin\theta\cos\theta) + I\dot{\phi}\sin\theta(\dot{\phi}\cos\theta + \dot{\psi}). \\ M_y &= I_0(\ddot{\phi}\sin\theta - 2\dot{\phi}\dot{\theta}\cos\theta) - I\dot{\theta}(\dot{\phi}\cos\theta + \dot{\psi}). \\ M_z &= I(\ddot{\psi} + \ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta). \end{aligned} \tag{3.9}$$

These equations are unsteady and non-linear. We can gain insight by examining the character of some special solutions and constants of the motion.

3.4 Tools of Chaotic System

The mathematical constructs that help to study a chaotic system are phase space, Poincare mapping and time series analysis. Phase space is the mathematical space of the dynamical variables of the system. Poincare map is the snapshot of phase space taken at regular time interval. The power spectrum is computed using Fourier analysis to display the frequency composition of the time variation of the dynamical variables. Here the study of the system was done using phase space and Poincare mapping.

3.4.1 Phase Space

Phase space is the mathematical space of the dynamical variables of a system. The phase space of a dynamical system is a mathematical space with orthogonal coordinate directions representing each of the variables needed to specify the instantaneous state of system. The 2N-dimensional space described by canonical coordinates of a dynamical system with N degrees of freedom is called phase space. A single point in this hypothetical space represents completely the state of the system. As the N-particle system evolves in the time according to the laws of motion, the representative point describes a trajectory or an orbit in the phase space. An important feature of the trajectory is that two trajectories corresponding to similar energies will pass very close to each other, but the phase trajectories will not intersect or self- intersect. This non crossing property derives from the fact that past and future states of a deterministic mechanical system are uniquely prescribed by the system state at a given time. Crossing of trajectories at a time would introduce ambiguity into past and future states, thereby rendering the system indeterminate.

According to Liouville's theorem a volume element in phase space cannot become smaller or larger a points within it evolves with time. For a system with one degree of freedom specified by a Hamiltonian $H(p, q)$,the equation $H(p, q) = 0$ determines the trajectory in (q, p)-space. All the trajectories are closed and the motion along any trajectory is periodic in time. So there is no chaos.

In the case of a system with two degrees of freedom where Hamiltonian is $H(q_1, p_1, q_2, p_2)$. The trajectory is determined by $H(q_1, p_1, q_2, p_2) = 0$, which is a 3d hypersurface.Here the two-dimensional tori is embedded in the three-dimensional energy shell(3d hypersurface). They divide the energy shell into an inside and outside. Thus if there was somehow a "gap" between tori,which occurs for non-integrable systems, then a trajectory in that gap cannot escape from it. The 2d torus is due to second integral I.

The absence of I,non integrable systems opens the gate to chaos. The trajectories cannot be confined to any tori. They can take any a much more complex form in the energy hypersurface.

3.4.2 Lyapunov exponent

A system is called chaotic if the system shows sensitive dependence to the initial condition. That is the neighboring orbits separate exponentially fast. This means that two trajectories stating very close together will diverge from each other and thereafter have totally different features (Strogatz, 2001).If $\delta(t)$ is the separation between trajectories, it grows as following in time.

$$\parallel \delta(t) \parallel \quad \parallel \delta_0 \parallel e^{\lambda t}. \tag{3.10}$$

where δ is the separation between the trajectories at first. The number λ is called Lyapunov exponent. A positive Lyapunov exponent is a signature of chaos.

$$\begin{aligned} \lambda &= \frac{1}{n} ln \mid \frac{\delta_n}{\delta_0} \mid . \\ &= \frac{1}{n} ln \mid \frac{f(x_0+\delta_0)-f(x_0)}{\delta_0} \mid . \\ &= \frac{1}{n} ln \mid (f)'(x_0) \mid . \end{aligned} \tag{3.11}$$

$$\lambda = \frac{1}{n}\sum_{i=0}^{n-1} ln \mid (f)'(x_i) \mid . \tag{3.12}$$

For stable fixed points and cycles,λ is negative and for chaotic systems it is positive.

3.4.3 Time Series Analysis

Power spectrum is useful tool in the study of dynamical systems. The power spectrum is computed using Fourier analysis to display the frequency composition of the time variation of the dynamical variables. The time evolution of a dynamical system is represented by the time variation f(t) or time series of its dynamical variables. Any function f(t) may be usefully represented as a superposition of periodic components. The determination of their relative strength is called spectral analysis . Depending upon the nature of the function we may represent it in two different but related ways . If f(t) is periodic ,then the spectrum may be expressed as a linear combination of oscillations whose frequencies are integer multiples of a basic frequency. This linear combination is called a Fourier series. However, it s more likely that f(t) is not periodic, and the spectrum must then be expressed in terms of oscillations with a continuum of frequencies. Such a spectral representation is called Fourier transform of (t).This representation is especially useful for chaotic dynamics. Because the Fourier transform is in general a complex–valued function it is often preferable to define a real-valued function which is the modulus square of the transform. This real function is called the power spectrum of f(t). The Fourier series representation of f(t) may be written compactly in complex notation

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{i\omega t}. \tag{3.13}$$

Where a_n are the amplitudes of the components of frequency $n\omega$.
The Fourier transform of f(t) can be written as

$$f(t) = \int_{-\infty}^{\infty} a(\omega) e^{i\omega t} d\omega. \tag{3.14}$$

where

$$a(\omega) = \frac{1}{2} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt. \tag{3.15}$$

As the Fourier transform often turns out to be complex, and it is useful to define a real-valued function,the power series spectrum as

$$S(\omega) = |a(\omega)|^2. \tag{3.16}$$

The power spectrum is the quantity typically calculated in experimental or numerical work. The power spectrum is introduced as a representation of the relative abundance of different frequencies in a given time series.

Chapter 4

Result and Discussion

4.1 Symmetric top

The symmetric top is the Lagrange's case of rigid body which has the inertia matrix

$$I = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_z \end{bmatrix}$$

That is $I_x = I_y \neq I_z$. The dynamics of the symmetric pendulum is studied using the Lagrange equation of motion. The Lagrangian $\mathbf{L} = T - V$,where T is the kinetic energy and V is the potential energy.

The kinetic energy of the system is given by

$$T = \frac{1}{2} I_x \omega_x^2 + \frac{1}{2} I_y \omega_y^2 + \frac{1}{2} I_z \omega_z^2. \tag{4.1}$$

As $I_x = I_y = I_1 \neq I_z$ the kinetic energy becomes

$$\begin{aligned} T &= \frac{1}{2} I_1(\omega_x^2 + \omega_y^2) + \frac{1}{2} I_z \omega_z^2. \\ &= \frac{1}{2} I_1(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{1}{2} I_3(\dot{\phi}\cos\theta + \dot{\psi}). \end{aligned} \tag{4.2}$$

The potential energy of the system is

$$V = Mgz\cos\theta. \tag{4.3}$$

where M is the mass of the body and z is the distance of the tip of the top from x axis. The Lagrangian $\mathbf{L}$ is defined as $\mathbf{L} = T - V$. Therefore from equation 4.2 and equation 4.3.

$$\mathbf{L} = \frac{1}{2} I_1(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{1}{2} I_3(\dot{\phi}\cos\theta + \dot{\psi}) - Mgz\cos\theta. \tag{4.4}$$

The equation of motion is obtained from Lagrange's equation.

$$\frac{d}{dt}\left(\frac{\partial \mathbf{L}}{\partial \dot{q_j}}\right) - \frac{\partial \mathbf{L}}{\partial q_j} = 0 \tag{4.5}$$

By solving the Lagrange's equation we get the behavior of the euler angles.

$$\begin{aligned} \ddot{\theta} &= \frac{1}{I_1}(\dot{\phi}\sin\theta\cos\theta(I_1 - I_z) - I_z\dot{\phi}\dot{\psi}\sin\theta + Mgz\sin\theta) \\ \ddot{\phi} &= \frac{2(I_z - I_1)\dot{\phi}\dot{\theta}\sin\theta\cos\theta - I_z\ddot{\psi}\cos\theta + I_z\dot{\psi}\dot{\theta}\sin\theta}{I_1\sin^2\theta + I_z\cos^2\theta} \\ \ddot{\psi} &= \dot{\phi}\dot{\theta}\sin\theta - \ddot{\phi}\cos\theta \end{aligned} \tag{4.6}$$

Using these equations we can get the information on the Euler angles. By using python the variation of θ, ϕ, ψ is determined.

In fig. 4.1 we can see the variation of $\theta, \phi, \psi, \dot{\theta}, \dot{\phi}$ and $\dot{\psi}$ with time. From

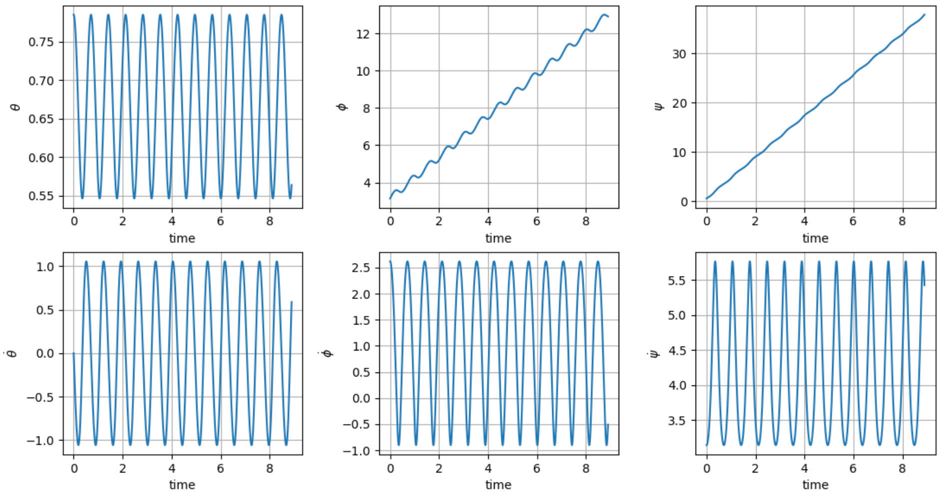


Figure 4.1: Variation of $\theta, \phi, \psi, \dot{\theta}, \dot{\phi}, \dot{\psi}$ with time.

these plots we can verify that symmetric top is definitely integrable system as the variation of the euler angles are periodic. The phase space of the system is plotted using θ and $\dot{\theta}$. It is given in fig. 4.2

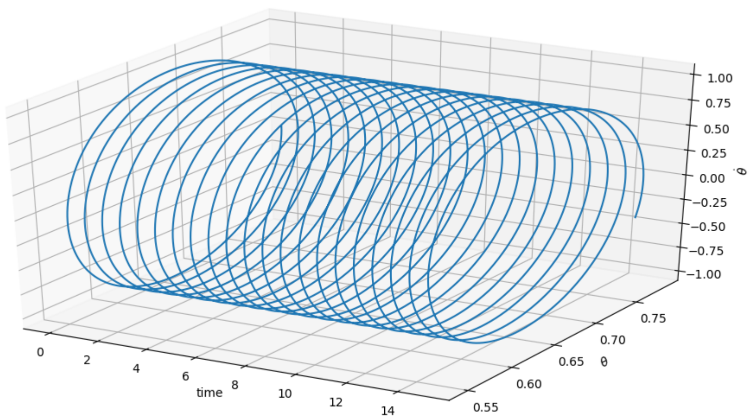


Figure 4.2: Phase space plot for θ

The tip of the top traces the motion of precession during the motion due to the overall effect of θ, ϕ and ψ. This motion can be described by plotting between the nutation angle and precession angle as in fig. 4.3.

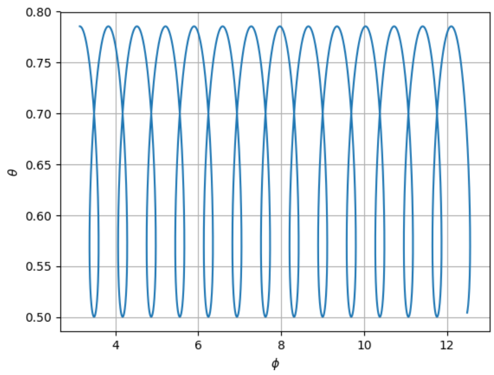


Figure 4.3: Precession of the top

From the time variation plots of euler angles and the phase space it is clear that the system is not showing any aperiodicity. For a top spinning top about an axis of symmetry the repeated returns will be regular and periodic.

4.2 Asymmetric top

When passing from the symmetric to the asymmetric top, one symmetry is broken, with the corresponding loss of one of the constants, we are left with a three degrees of freedom system with only two constants. At this point one must check whether the third constant is available or if the system shows chaotic behavior. It is at this point that the numerical indicators of chaos are a most useful tool which can confirm the existence or nonexistence of the missing integral, clarifying thus the integrability (or non-integrability) of the top(Barrientos et al., 1995).

A symmetric top is turned into asymmetric by making the system having three different moment of inertia. The inertia matrix is changed to,

$$I = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_1(1+\epsilon) & 0 \\ 0 & 0 & I_z \end{bmatrix}$$

Due to the change in inertia matrix the Lagrange's equation is thus modified as follows,

$$\begin{aligned} \mathbf{L} = \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{1}{2}I_3(\dot{\phi}\cos\theta + \dot{\psi}) - Mgz\cos\theta + \\ \frac{1}{2}\epsilon I_1(\dot{\phi}\cos\psi\sin\theta - \dot{\theta}\sin\psi)^2. \end{aligned} \tag{4.7}$$

With this $\mathbf{L}$ the Lagrange's equations becomes,

$$\begin{aligned} \ddot{\theta} = & \frac{1}{I_1 + \epsilon I_1 \sin^2\psi}(\dot{\phi}\sin\theta\cos\theta(I_1 - I_z) - I_z\dot{\phi}\dot{\psi}\sin\theta + Mgz\sin\theta \\ & + \frac{1}{2}\epsilon I_1 \sin 2\psi(\ddot{\phi}\sin\theta + \dot{\theta}\dot{\phi}\cos\theta) + \epsilon I_1\dot{\phi}\dot{\psi}\sin\theta\cos 2\psi \\ & - \epsilon\dot{\theta}\dot{\psi}\sin 2\psi + \epsilon I_1\dot{\phi}\cos\psi\cos\theta(\dot{\phi}\cos\psi\sin\theta - \dot{\theta}\sin\psi). \end{aligned} \tag{4.8}$$

$$\ddot{\phi} = \frac{1}{I_1 \sin^2\theta + I_z \cos^2\theta + \epsilon I_1 \sin^2\theta\cos^2\psi}(2(I_z - I_1)\dot{\phi}\dot{\theta}\sin\theta\cos\theta$$
$$- I_z\ddot{\psi}\cos\theta + I_z\dot{\psi}\dot{\theta}\sin\theta + \epsilon I_1\dot{\phi}\dot{\psi}\sin^2\theta\sin 2\psi + 0.5\epsilon I_1\ddot{\theta}\sin 2\psi\sin\theta \tag{4.9}$$
$$- \epsilon I_1\dot{\phi}\dot{\theta}\sin 2\theta\cos^2\psi + 2\epsilon I_1\dot{\theta}\dot{\psi}\cos 2\psi\sin\theta + 0.5\epsilon I_1\dot{\theta}^2\sin 2\psi\cos\theta).$$

$$\ddot{\psi} = \frac{\dot{\phi}\dot{\theta}\sin\theta - \ddot{\phi}\cos\theta - \epsilon I_1(\dot{\phi}\cos\psi\sin\theta - \dot{\theta}\sin\psi)(\dot{\phi}\sin\psi\sin\theta + \dot{\theta}\cos\psi)}{I_z}. \tag{4.10}$$

Using these three equations the variation of euler angle is found for different ϵ. The variation is shown below in fig. 4.4. From this figure it is clear that the euler angle propagation has being disturbed due to the asymmetric property.

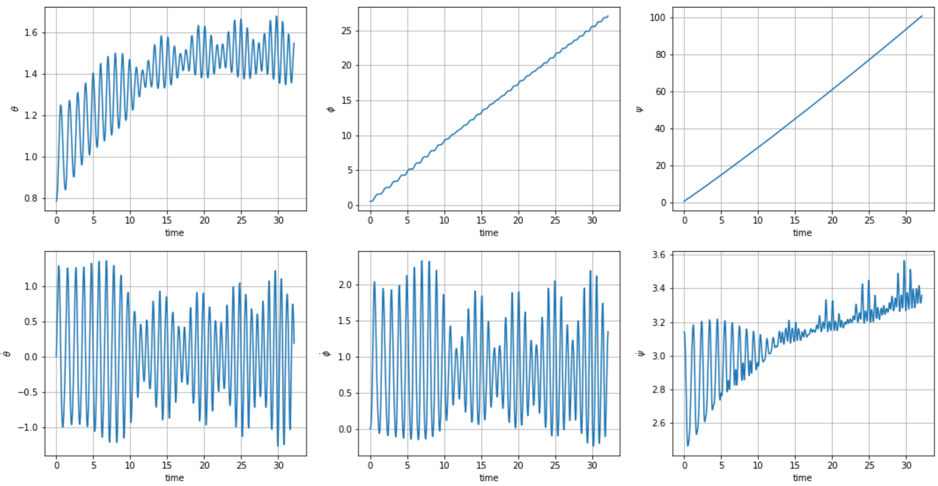
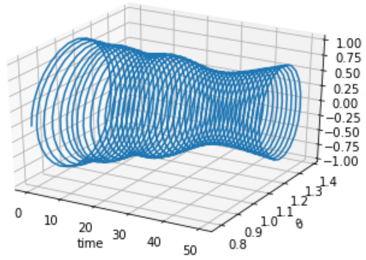
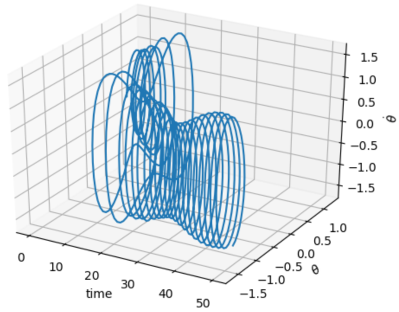


Figure 4.4: Variation of $\theta, \phi, \psi, \dot{\theta}, \dot{\phi}, \dot{\psi}$ with time.

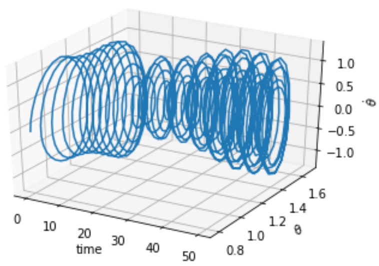
To detect if this disturbance is due to the non- integrability or chaotic behavior it is studied through the indicators of chaos, such as the power spectrum, Lyapunov exponents and phase space maps.These are the techniques we will employ for the analysis of the asymmetric heavy top.



(a) Phase space trajectory for $\epsilon = -0.02$



(b) Phase space trajectory for $\epsilon = 0.08$

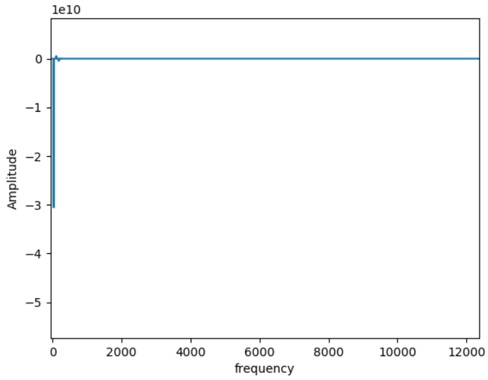


(c) Phase space trajectory for $\epsilon = 0.23$

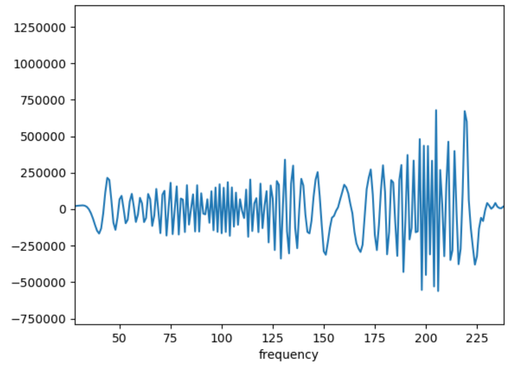
Figure 4.5: Variation of phase space for different ϵ

From fig. 4.5 we can see that the phase space is no more periodic and it changes as we change the value of *epsilon*. The aperiodic behavior off phase space is an indicator for non-integrability.

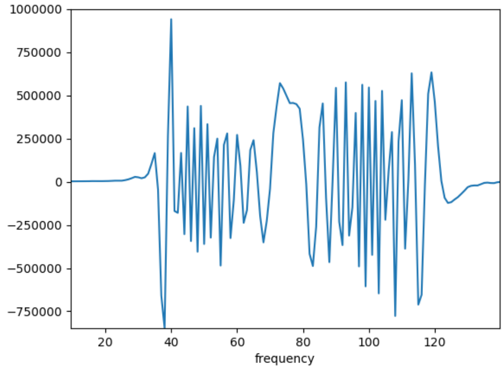
The fig. 4.6ref shows the power spectrum for $\epsilon = 0.04$ and $\epsilon = 0.14$. This is the lower frequency region. For symmetric top fig. 4.6(a)the spectrum shows no peaks which shows the periodicity in the motion. Whereas the power spectrum for $\epsilon = 0.04$ and $\epsilon = 0.14$ gives more peaks in the low frequency region which depicts the chaotic behavior.



(a) Power spectrum for symmetric top



(b) Power spectrum for $\epsilon = 0.04$



(c) Power spectrum for $\epsilon = 0.14$

Figure 4.6: The power spectrum for different ϵ

Lyapunov exponent is one of the strongest indicator of non-integrability. The Lyapunov exponent for the variation of time evolution due to two trajectories is plotted in fig. 4.7. For stable fixed points and cycles,λ, Lyapunov exponent is negative and for chaotic systems it is positive. Here in this plot we can see that at first he exponent goes negative but as the trajectories starts deviating the λ turns positive and it is in positive state for rest of the time.

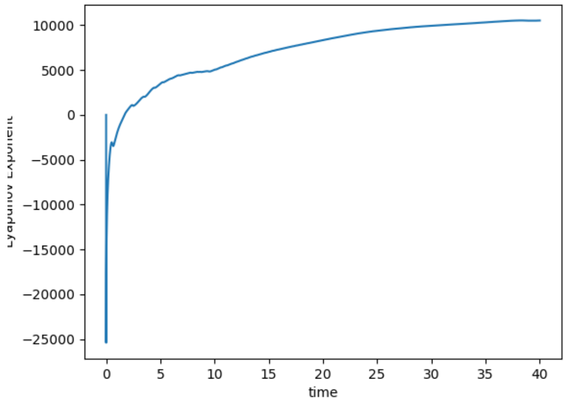


Figure 4.7: Lyapunov exponent variation for $\epsilon = 0.04$

From all these analysis it is clear that the system has turned non-integrable. This again gives out that there is no existence of a third constant of integral which makes the system integrable. The breaking of symmetry of the Lagrange's top thus leads to a non- integrable system.

4.3 Symmetric top with harmonic stimulus

The next case is of a symmetric top mounted on a vibrating base. The motion of this system is described by Euler's angles θ, ϕ and ψ. The system is excited by harmonic forces $l_1 \sin\omega_1 t + l_2 \sin\omega_2 t$. The Lagrangian of the system is given below,

$$\begin{aligned}\mathbf{L} = \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{1}{2}I_3(\dot{\phi}\cos\theta + \dot{\psi}) \\ -Mg(z + l_1\sin\omega_1 t + l_2\sin\omega_2 t)\cos\theta.\end{aligned} \tag{4.11}$$

where l_1 and l_2 are the amplitude of external excitation disturbance and $\omega 1$ and ω_2 are the frequencies of external excitation disturbances. With this $\mathbf{L}$ the Lagrange's equations becomes,

$$\begin{aligned}\ddot{\theta} &= \frac{1}{I_1}(\dot{\phi}\sin\theta\cos\theta(I_1 - I_z) - I_z\dot{\phi}\dot{\psi}\sin\theta \\ &+ Mg(z + l_1\sin\omega_1 t + l_2\sin\omega_2 t)\sin\theta) \\ \ddot{\phi} &= \frac{2(I_z - I_1)\dot{\phi}\dot{\theta}\sin\theta\cos\theta - I_z\ddot{\psi}\cos\theta + I_z\dot{\psi}\dot{\theta}\sin\theta}{I_1\sin^2\theta + I_z\cos^2\theta} \\ \ddot{\psi} &= \dot{\phi}\dot{\theta}\sin\theta - \ddot{\phi}\cos\theta\end{aligned} \tag{4.12}$$

As we can see there is no change in $p\ddot{h}i$ and $\ddot{\psi}$ from the case of symmetric top to study the properties of the system we need to take only the case of θ. Since ϕ and ψ are absent in the Lagrangian we get first two integrals of motion expressing the conjugate momenta. The momentum integrals are,

$$\begin{aligned}P_\phi = \frac{\partial\mathbf{L}}{\partial\dot{\phi}} = \beta_\phi. \\ P_\psi = \frac{\partial\mathbf{L}}{\partial\dot{\psi}} = \beta_\psi.\end{aligned} \tag{4.13}$$

As the system is having three variables we use the Routh's procedure to reduce the system into one variable dependence.The Routhian is defined as

$$
\begin{aligned}
R &= \mathbf{L} - \beta_\phi \dot{\phi} - \beta_\psi \dot{\psi} \\
&= \frac{1}{2} I_1 \dot{\theta}^2 - \left(\frac{(\beta_\phi - \beta_\psi \cos\theta)^2}{2 I_2 \sin^2\theta} + \frac{\beta_\phi^2}{2 I_z} + Mg(z + l_1 \sin\omega_1 t + l_2 \sin\omega_2 t)\cos\theta\right).
\end{aligned}
\tag{4.14}
$$

With Routhian we can deduce the equation of motion in one variable by,

$$
\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{\theta}}\right) - \frac{\partial R}{\partial \theta} = 0 \tag{4.15}
$$

This gives ,

$$
\ddot{\theta} = \frac{Mg}{I_1}(z + l_1 \sin\omega_1 t + l_2 \sin\omega_2 t)\sin\theta) - \frac{\beta_\phi^2 (1 - \cos\theta)^2}{I_1^2 \sin^3\theta} \tag{4.16}
$$

Using the equation 4.12 we can plot the variation of Euler angles with time.

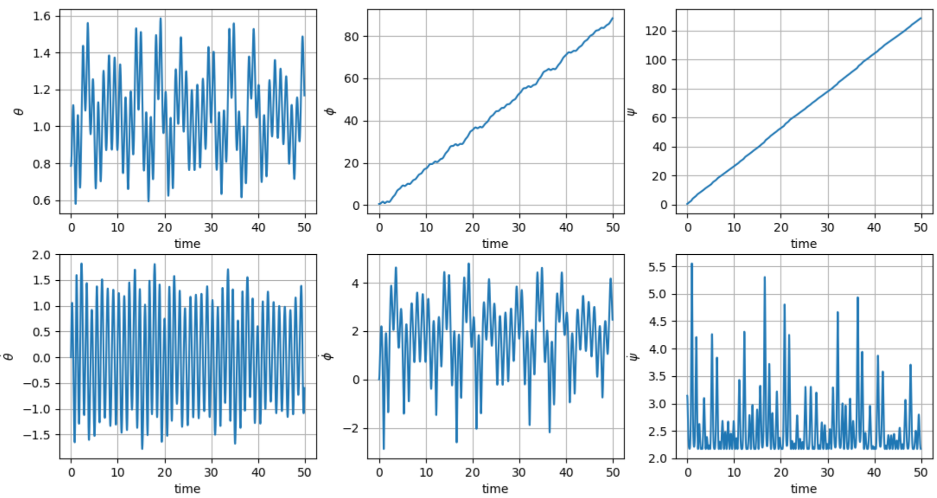


Figure 4.8: The variation of euler angles

From the fig. 4.8 we can deduce that the non-periodic behavior of the system. Here we used the parameters $l_1 = 0.4m, l_2 = 0.6m, \omega_1 = 1.59rads^{-1}$ and $\omega_2 = 1.24rads^{-1}$. To verify the non-integrability the chaotic indicators are used. The phase space of the system for the above parameters is given by fig. 4.9. This obtained from the reduced equation of motion using Routhian.

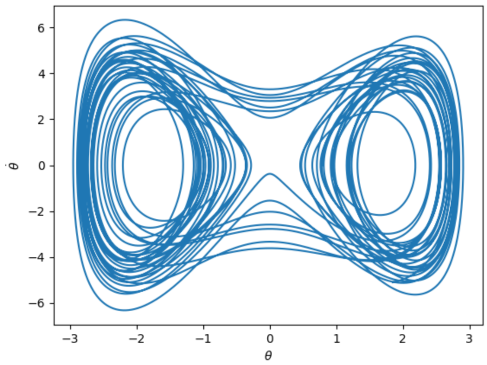


Figure 4.9: Phase space for a particular θ

Since only one variable is present we can show the sensitivity to initial condition much clearly as in fig. 4.10.

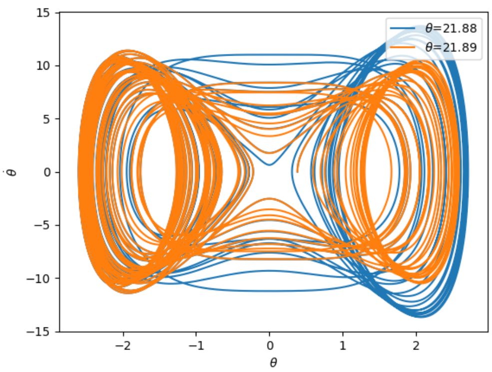


Figure 4.10: Phase space for two close initial conditions θ

The power spectrum and Lyapunov exponent variation was done for the system. The power spectrum is given in fig. 4.11. In these we can see may peaks denoting the non linearity of the system.

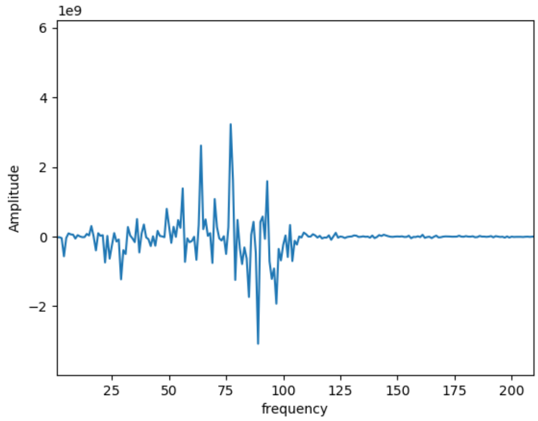


Figure 4.11: The power spectrum

The Lyapunov exponent variation is given in fig. 4.12. The λ turns positive indicating the deviation in the trajectories and hence the chaotic property of the system.

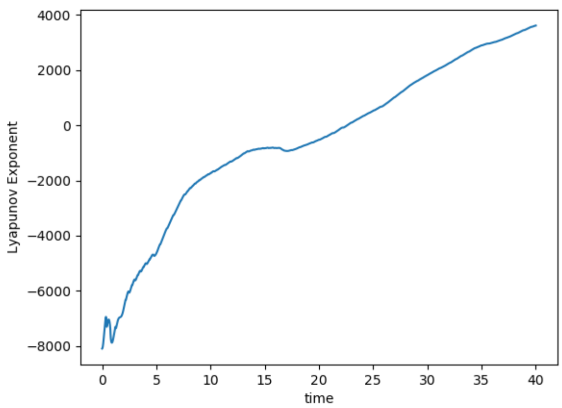


Figure 4.12: The Lyapunov exponent variation

Hence the symmetric top with external harmonic stimulus is proved to be non-integrable.

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