A CONFORMAL FIELD THEORY APPROACH TO CRITICAL SYSTEMS

Project Report II

submitted in partial fulfillment for the degree of

MASTER OF SCIENCE

IN

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by

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Under the Guidance of

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DECLARATION

I hereby declare that the report of the P.G. project work entitled "A CONFORMAL FIELD THEORY APPROACH TO CRITICAL SYSTEMS" which is submitted to National Institute of Technology Karnataka, Surathkal, in partial fulfillment of the requirement for the award of the Degree of Master of Science in the Department of Physics, is a bonafide report of the work carried out by me. The material contained in this report has not been submitted to any University or Institution for the awards of any degree. In keeping with the general practice in reporting scientific observations, due acknowledgement has been made whenever the work described is based on the findings of other investigators.

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CERTIFICATE

This is to certify that the project entitled "A CONFORMAL FIELD THEORY APPROACH TO CRITICAL SYSTEMS" is an authenticated record of work carried out by Anasuya G Nair, Reg. No 186020 in partial fulfillment of the requirement for the award of the Degree of Master of Science in Physics which is submitted to the Department of Physics, National Institute of Technology, Karnataka during the period 2018-2020.

Dr Deepak Vaid
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ABSTRACT

Conformal field theories have been at the centre of much attention for the last 15 years due to their relevance in areas of modern theoretical physics such as string theory, quantum gravity, condensed matter physics and quantum field theories. They are Euclidean quantum field theories that are characterised by their symmetry group which involves local conformal transformations along with Euclidean symmetries. As such, it can be considered as a prototype example for a constructive interplay between mathematics and physics. Compared to ordinary quantum field theories, conformal field theories in two dimensions are highly constraining and can be defined in a rather abstract way via operator algebras and their representation theory. This project is an introductory study of conformal field theories and AdS/CFT correspondence. It is an attempt to solve critical systems, namely the critical 2D Ising model and the holographic superconductors, where the critical exponents are determined using the tools of 2D conformal field theories and AdS/CFT.

Keywords— central charge, Virasoro algebra, conformal bootstrap, Kac determinant, holographic principle, large-N, supergravity.

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1 Introduction

A phase transition in a system is the change in the state of the system from one state to another which exhibits remarkable differences in its properties from the initial state. A simple example of a phase transition is the condensation of water. Phase transitions are affected by external parameters such as temperature, pressure, etc. They can be of two types: first order phase transitions (entropy discontinuous across the boundary of the two systems) and second order phase transitions (entropy continuous across the boundary). The points at which a phase transition occurs are called critical points. The first order phase boundary becomes second order at the critical point. The associated phenomena are known as critical phenomena and the systems are said to be critical systems. A graphical representation of the stable phase of a system for every combination of macroscopic variables is called its phase diagram.

A phase transition involves a symmetry breaking process. The high temperature phase usually contains more symmetries than the low temperature phase due to spontaneous symmetry breaking. A measure of the degree of order across the boundaries of the system during the transition is given by an order parameter. It is zero in one phase and non-zero in the other. A phase transition is closely related to a quantity called the correlation length. It is the characteristic length scale at which the overall properties of the parts of a system begin to differ i.e. at this scale, the fluctuations of the microscopic degrees of freedom are correlated. At the critical point, the correlation length is large compared to the microscopic scale of the system. Critical phenomena are characterized by a divergent susceptibility, an infinite correlation length, and a power law decay of correlations near criticality. Such divergences are parametrized by critical exponents.

In a quantum field theory, the basic objects we compute are the correlation functions. It is a conditional probability that gives the measure of the order of a system. A 2-point correlation function is the average product of the two states and is given by

$$\langle \sigma_1 \sigma_2 \rangle = \frac{1}{|r_{12}|^{2\triangle}} \tag{1.1}$$

where $\triangle$ is called the scaling or conformal dimension, which will be discussed later on. The divergence of the correlation length in the vicinity of a second order phase transition implies that the properties near a critical point can be described within an effective theory. This project aims at determining the critical exponents of systems undergoing a second order phase transition using conformal field theories.

1.1 The Ising Model

A well known example of a phase transition is the Ising model, which describes ferromagnetism in statistical physics. The external parameters characterising the transition are temperature and magnetization. By changing the magnetic field at a constant temperature which is less than a critical (Curie) temperature, a sample undergoes a first order phase transition from a spin-up state, in which the average magnetisation is positive, to a spin-down state, in which the average is negative. By changing the temperature at a fixed zero magnetic field, the system undergoes a second order phase transition at the Curie temperature. If the temperature is below the Curie temperature, the atoms are aligned parallel with respect to each other, which causes spontaneous magnetization and the material is ferromagnetic. Above the Curie temperature, spontaneous symmetry breaking occurs and the atoms lose their ordered magnetic moments thus turning the material paramagnetic.

In Ising model, we consider a 2D array of spins ± 1 at some temperature. If there are N such identical spin$-1/2$ particles in a lattice kept in the presence of a magnetic field H, the total energy of the system is given by

$$E = -J \sum_{i,j} \sigma_i \sigma_j - \mu H \sum_{i=1}^{N} \sigma_i \tag{1.2}$$

where σ_i are the spins, μ is the dipole moment, the sum is over nearest neighbour pairs and J is a coupling constant. The order parameter is the magnetization. It changes discontinuously below the Curie temperature and continuously above the Curie temperature. In a second order phase transition, the discontinuity in the second derivative of the Gibbs free energy leads to a discontinuity in the response functions. Critical exponents tell us how these response functions change.

1.2 Conformal Symmetry

Before going any further, I would like to introduce a few important terms that one should be familiar with:

- **Projective Space**: A space of one dimensional subspaces of a given vector space.
- **Compactification**: Embedding of a topological space as a dense subset of a compact space (to refrain points from going off to infinity).
- **Pseudo-Riemannian Manifold**: A differentiable manifold with a smooth tensor field which assigns for each point, a non-degenerate and symmetric bilinear form on the tangent space. An example is the Lorentzian manifold with $(p, q) = (n - 1, 1)$ *or* $(1, n - 1)$.
- **Projective Representation of a group** on a vector space over a field: A group homomorphism from the group to the projective linear group. For example, the irrep of SU(2) in an odd dimension (eg: integer spin) descends to SO(3), while in an even dimension (eg: half-integer spin) descends to the projective representation of SO(3). This is called the spinorial representation or representation of spin groups.

Now, let us define a conformal transformation. It is a transformation in the complex plane given by

$$w = f(z) \tag{1.3}$$

which preserves local angles and orientations. Here, z is called the dynamical critical exponent. In relativistic field theories, this constant is set to be 1 since the space and time coordinates are on equal footing. An example of such a map is the Poincaré transformation in flat spacetime

$$x^\mu = \Lambda^\mu_\nu x^\nu + a^\mu \tag{1.4}$$

where Λ parametrizes Lorentz transformation and a^μ parametrizes translations. In other words, Poincaré transformations are the isometries of flat spacetime.

A conformal map is thus a diffeomorphism between two Riemannian manifolds. Two Riemannian metrics g and h are said to be conformally equivalent if $g = \Omega h$, where Ω is the conformal factor. It can be considered as a local rescaling of the metric. Another familiar example of a conformal transformation is a stereographic projection $\pi = S^2 \to \mathbb{R}^2$ given by

$$(x, y, z) = \frac{1}{1-z}(x, y) \tag{1.5}$$

where $\dfrac{1}{1-z}$ is the conformal factor. A conformal field theory is a quantum field theory

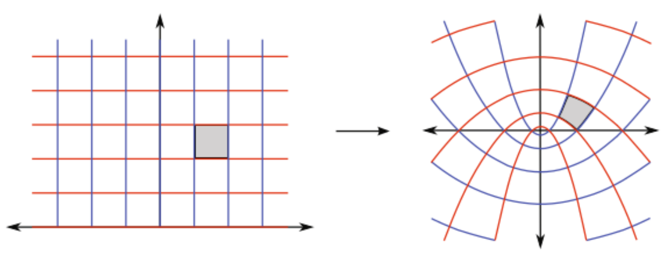

Figure 1: Conformal Transformation in 2D

which remains invariant under a conformal transformation i.e. the set of coordinate transformations that leave the metric invariant upto an overall function. Classically, it means that the equations of motion are invariant. In quantum mechanics, it yields a projective unitary representation of the group (Wigner's Theorem). When $\Lambda(x) = 1$, the conformal symmetry reduces to the Lorentz symmetry.

Conformal invariance is not equivalent to scale invariance by definition. But, there is a hidden symmetry in nature that allows us to equate the two in ordinary quantum field theories. In 3 or more dimensions, conformal invariance does not turn out to give much more information than ordinary scale invariance. But in 2D, the conformal algebra becomes infinite dimensional, leading to significant restrictions on the 2D conformally invariant theories. As a result, we can find the correlation functions of many theories without the need for computing the Lagrangian. An important application of this is in the statistical models. In 2D statistical models, the continuum description at a second order phase transition is given by a conformal field theory. The 2D Ising model is a non-trivial many body problem that is exactly solvable and shows a phase transition. There is a duality that relates the two phases and there exists a second order phase transition at the self-dual point (Curie temperature), where the typical configurations have fluctuations on all length scales. Thus, the field theory describing the model at its critical point should be scale invariant.

Another important application is in string theory, where the CFT is a 2D field theory living on the world volume of a string moving in a spacetime i.e. the movement of fundamental strings can be described by a suitable 2D CFT. One can say that CFT is the space of classical solutions to the spacetime equations of string theory. It helps us to solve the theory exactly to all orders in perturbation theory and we can sum all the contributions of the so-called worldsheet instantons (a classical solution to equations of motion with a finite, non-zero action). Thus, CFT is a very powerful tool for string theory.

2 Conformal Transformations in d-Dimensions

In this section, we will discuss about the constraints imposed by conformal invariance in d-dimensions. Let $\mathcal{M} = \mathbb{R}^{p,q}$ be our manifold where $d = p + q$ and $p, q \in \mathbb{Z}$. The Riemannian metric is given by

$$g_{\mu\nu} = diag(1, 1...(p - times), -1, -1, ...(q - times))$$

We consider a fixed background metric so that the transformation is an honest, physical symmetry. This is the situation in quantum field theory. Let us consider a smooth change of coordinates $X \to X'$. A conformal map is given by

$$g_{\mu\nu} \mapsto g'_{\mu\nu} = \frac{\partial x^{\alpha\prime}}{\partial x^{\mu}} \frac{\partial x^{\beta\prime}}{\partial x^{\nu}} g_{\alpha\beta} = \Lambda(x) g_{\mu\nu} \tag{2.1}$$

where $\Lambda(x)$ is the scale factor. Note that we use the Einstein summation convention here. Thus, conformal transformations are coordinate transformations that leaves the metric invariant up to a scale change.

If U^{μ} and V^{ν} are two tangent vectors, then the angle between them is given by

$$\theta = \frac{g_{\mu\nu} U^{\mu} V^{\nu}}{\sqrt{(g_{\mu\nu} U^{\mu} V^{\nu})^2}} \tag{2.2}$$

A group of such transformations is called a conformal group i.e. it is the connected component containing the identity transformation in the group of all conformal transformations of $\mathcal{M}$. Any manifold has a conformal group as long as its metric obeys Eq. (2.1). For flat spaces, $\Lambda(x) = 1$ corresponds to the Poincaré group which includes translations and Lorentz rotations. While, $\Lambda(x) =$ constant corresponds to scale transformations (dilatations). The conformal group is basically a Lie group in higher dimensions, but not quite so in lower dimensions (eg: d = 2), as we will see later on.

2.1 Conditions for Conformal Invariance

Let us consider an infinitesimal conformal transformation [18] to the first order

$$x'^{\mu} = x^{\mu} + \varepsilon^{\mu}(x) + \mathcal{O}(\varepsilon^2) \tag{2.3}$$

Substituting this in Eq. (2.1), we get

$$\begin{aligned} g'_{\mu\nu} &= g_{\alpha\beta} \frac{\partial(x^{\alpha} + \varepsilon^{\alpha}(x) + \mathcal{O}(\varepsilon^2)}{\partial x^{\mu}} \frac{\partial(x^{\beta} + \varepsilon^{\beta}(x) + \mathcal{O}(\varepsilon^2)}{\partial x^{\nu}} \\ &= g_{\alpha\beta} \left(\delta^{\alpha}_{\mu} + \frac{\partial \varepsilon^{\alpha}}{\partial x^{\mu}} + \mathcal{O}(\varepsilon^2) \right) \left(\delta^{\beta}_{\nu} + \frac{\partial \varepsilon^{\beta}}{\partial x^{\nu}} + \mathcal{O}(\varepsilon^2) \right) \\ &= g_{\mu\nu} + \left(\frac{\partial \varepsilon_{\mu}}{\partial x^{\nu}} + \frac{\partial \varepsilon_{\nu}}{\partial x^{\mu}} \right) + \mathcal{O}(\varepsilon^2) \end{aligned}$$

In order to satisfy Eq. (2.1), we require that

$$\partial_{\mu}\varepsilon_{\nu} + \partial_{\nu}\varepsilon_{\mu} = f(x) g_{\mu\nu} \tag{2.4}$$

where $\partial_{\mu} = \frac{\partial}{\partial x^{\mu}}$ and $f(x)$ is called the conformal Killing factor. This is called the conformal Killing equation. Multiplying both sides of Eq. (2.4) by $g^{\mu\nu}$ and taking the trace, we get

$$2\partial^{\mu}\varepsilon_{\mu} = f(x)d \tag{2.5}$$

Substituting it back to Eq. (2.4), we get

$$\partial_{\mu}\varepsilon_{\nu} + \partial_{\nu}\varepsilon_{\mu} = \frac{2}{d}(\partial.\varepsilon) g_{\mu\nu} \tag{2.6}$$

where $\partial.\varepsilon = \partial^{\mu}\varepsilon_{\mu}$. Therefore, the conformal factor for this infinitesimal transformation is given by

$$\Lambda(x) = 1 + \frac{2}{d}(\partial.\varepsilon) + \mathcal{O}(\varepsilon^2) \tag{2.7}$$

Modifying Eq. (2.6) by acting ∂^{ν} on both sides,

$$\partial^{\nu}\partial_{\mu}\varepsilon_{\nu} + \partial^{\nu}\partial_{\nu}\varepsilon_{\mu} = \frac{2}{d}\partial^{\nu}(\partial.\varepsilon) g_{\mu\nu}$$

$$\partial_\mu(\partial.\varepsilon) + \Box\,\varepsilon_\mu = \frac{2}{d}\partial_\mu(\partial.\varepsilon) \tag{2.8}$$

where $\Box = \partial^\mu\partial_\mu$ is the d'Alembertian operator. Acting ∂_ν on Eq. (2.8), we get

$$\partial_\mu\partial_\nu(\partial.\varepsilon) + \Box\,\partial_\nu\varepsilon_\mu = \frac{2}{d}\partial_\mu\partial_\nu(\partial.\varepsilon) \tag{2.9}$$

Interchanging μ and ν since they are dummy indices and adding the resultant expression to Eq. (2.9),

$$(\partial_\mu\partial_\nu + \partial_\nu\partial_\mu)(\partial.\varepsilon) + \Box\,(\partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu) = \frac{2}{d}(\partial_\mu\partial_\nu + \partial_\nu\partial_\mu)(\partial.\varepsilon)$$

$$\left(1 - \frac{2}{d}\right)(\partial_\mu\partial_\nu + \partial_\nu\partial_\mu)(\partial.\varepsilon) + \Box\,\left(\frac{2}{d}(\partial.\varepsilon)g_{\mu\nu}\right) = 0$$

The above step used the result in Eq. (2.6). On further simplification,

$$((d-2)\partial_\mu\partial_\nu + g_{\mu\nu}\Box)(\partial.\varepsilon) = 0 \tag{2.10}$$

Contracting with $g^{\mu\nu}$,

$$(d\,\partial^\mu\partial_\mu - 2\partial^\mu\partial_\mu + \Box)(\partial.\varepsilon) = 0$$

$$(d-1)\,\Box\,(\partial.\varepsilon) = 0 \tag{2.11}$$

Eq. (2.11) gives the condition for conformal invariance. A finite number of parameters, given by $\frac{1}{2}(d+1)(d+2)$, are needed to specify a conformal transformation in d-dimensions [18]. In 2D, the first term in Eq. (2.10) vanishes and the parameters specifying the transformation is well-defined everywhere. The algebra of infinitesimal conformal transformations is infinite dimensional as there is an infinite number of local transformations which are equivalent to local dilatations. Since a local analytic function provides a conformal mapping, the number of parameters specifying the transformation is infinite. Thus, a 2D CFT is very strongly constrained by its symmetries.

Another useful relation can be found by acting ∂_α on Eq. (2.6) and permuting the indices

$$\partial_\alpha\partial_\mu\varepsilon_\nu + \partial_\alpha\partial_\nu\varepsilon_\mu = \frac{2}{d}g_{\mu\nu}\partial_\alpha(\partial.\varepsilon)$$

$$\partial_\nu\partial_\alpha\varepsilon_\mu + \partial_\mu\partial_\alpha\varepsilon_\nu = \frac{2}{d}g_{\alpha\mu}\partial_\nu(\partial.\varepsilon)$$

$$\partial_\mu\partial_\nu\varepsilon_\alpha + \partial_\nu\partial_\mu\varepsilon_\alpha = \frac{2}{d}g_{\nu\alpha}\partial_\mu(\partial.\varepsilon)$$

Adding the last two equations and subtracting the first equation from it, we get

$$2\partial_\mu\partial_\nu\varepsilon_\alpha = \frac{2}{d}(-g_{\mu\nu}\partial_\alpha + g_{\alpha\mu}\partial_\nu + g_{\nu\alpha}\partial_\mu)(\partial.\varepsilon) \tag{2.12}$$

2.2 Conformal Algebra in $d > 2$ Dimensions

From Eq. (2.11), we see that for $d > 2$, the third derivative of ε is required to be zero. Therefore, ε is atmost quadratic in nature.

$$\varepsilon = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho \tag{2.13}$$

The coefficients $a_\mu, b_{\mu\nu}, c_{\mu\nu\rho} \ll 1$ and $c_{\mu\nu\rho}$ is symmetric in the last two indices. Let us consider the constraints separately since the constraints for conformal invariance are independent of x^μ.

Case 1: For zeroth order in x

$\varepsilon = a_\mu$ is a constant term corresponding to infinitesimal translations $x^{\mu\prime} = x^\mu + a^\mu$, for which the corresponding generator is the momentum operator $P_\mu = -i\partial_\mu$

Case 2: For linear dependence in x

Substituting $\varepsilon = b_{\mu\nu}x^\nu$ in Eq. (2.6), we get

$$\partial_\mu b_{\nu\mu}x^\mu + \partial_\nu b_{\mu\nu}x^\nu = \frac{2}{d}(g^{\alpha\beta}b_{\alpha\beta})g_{\mu\nu}$$

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d}(g^{\alpha\beta} b_{\alpha\beta}) g_{\mu\nu} \tag{2.14}$$

This implies the symmetric part of b is proportional to the metric. Therefore, we can write

$$b_{\mu\nu} = \lambda g_{\mu\nu} + m_{\mu\nu} \tag{2.15}$$

where m is the asymmetric part that yields the infinitesimal Lorentz rotations

$$x^{\mu\prime} = (\delta^\mu_\nu + m^\mu_\nu) x^\nu$$

for which the corresponding generator is the angular momentum operator $L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$. Meanwhile, $\lambda g_{\mu\nu}$ is the symmetric part corresponding to the infinitesimal scale transformations

$$x^{\mu\prime} = (1 + \lambda) x^\mu$$

for which the corresponding generator is the dilatation operator $D = -i x^\mu \partial_\mu$.

Case 3: For quadratic dependence in x

Substituting Eq. (2.13) in Eq. (2.12), we get

$$\partial.\varepsilon = b^\mu_\mu + 2c^\mu_{\mu\rho} x^\rho \tag{2.16}$$

This implies $\partial_\nu(\partial.\varepsilon) = 2c^\mu_{\mu\rho}$

Therefore,

$$c_{\mu\nu\rho} = g_{\mu\rho} b_\nu + g_{\mu\nu} b_\rho - g_{\nu\rho} b_\mu \tag{2.17}$$

where $b_\mu = \frac{1}{d}c^\rho_{\rho\mu}$

$$\begin{aligned} c_{\mu\nu\rho}x^\nu x^\rho &= (g_{\mu\rho}b_\nu + g_{\mu\nu}b_\rho - g_{\nu\rho}b_\mu)x^\nu x^\rho \\ &= b_\nu x^\nu x_\mu + b_\rho x_\mu x^\rho - b_\mu x_\nu x^\nu \\ &= (b_\nu x^\nu + b_\rho x^\rho)x_\mu - b_\mu x_\nu x^\nu \\ &= 2(b.x)x_\mu - b_\mu(x.x) \end{aligned}$$

where x is a conformal Killing field. Therefore, the infinitesimal conformal transformation is given by

$$x^{\mu\prime} = x^\mu + 2(b.x)x^\mu - (x.x)b^\mu \tag{2.18}$$

Such transformations are called special conformal transformations, for which the corresponding infinitesimal generator is $K_\mu = -i(2x_\mu x^\nu\partial_\nu - (x.x)\partial_\mu)$. The commutation relations of the operators are given in [Appendix A]. The finite transformation of special conformal transformations is given by

$$x^{\mu\prime} = \frac{x^\mu - (x.x)b^\mu}{1 - 2(b.x) + (b.b)(x.x)} \tag{2.19}$$

On expanding the denominator for small b^μ, one can get the infinitesimal form in Eq. (2.18). Furthermore, the conformal factor of special conformal transformations is given by

$$\Lambda(x) = (1 - 2(b.x) + (b.b)(x.x))^2 \tag{2.20}$$

A special conformal transformation can be thought of as an inversion of x, followed by a translation, which is again followed by an inversion. But, inversion is a discrete transformation. We are interested in continuous transformations. In physics, unlike rotations and translations of Poincaré symmetry, an object cannot be physically transformed by an inversion symmetry. Thus, we study theories that are invariant under this symmetry (hidden symmetry) such as gauge symmetry and general covariance (diffeomorphism covarience).

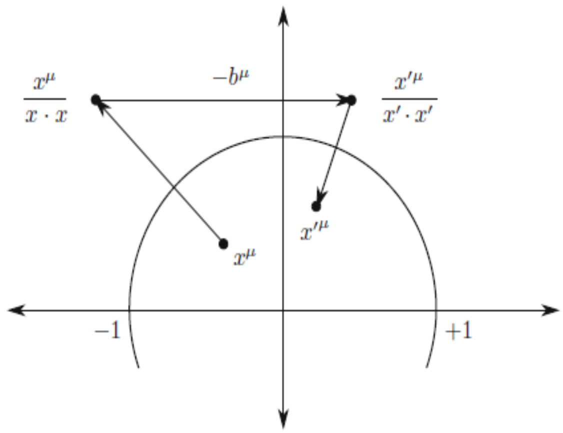

Figure 2: Illustration of a finite special conformal transformation

2.3 Conformal Algebra in 2-Dimensions

It is a quantum field theory on 2D Euclidean space invariant under local conformal transformations. A 2D conformal field theory, in comparison to ordinary quantum field theories, can be defined in an abstract way via operator algebras and their representation theory. From Eq. (2.11), the restriction that $\varepsilon(x)$ is of atmost second order in x does not apply here. It is infinite dimensional and hence highly constraining. From Eq. (2.6), the condition for invariance under infinitesimal conformal transformations in 2D becomes the Cauchy Riemann equations $\partial_0\varepsilon_0 = \partial_1\varepsilon_1$ and $\partial_0\varepsilon_1 = -\partial_1\varepsilon_0$.

Thus, we consider a complex function that satisfies the above conditions. Such functions are called holomorphic (in some open set). For a change of coordinates $z = x_0 + ix_1$ and $\bar{z} = x_0 - ix_1$, we can write $\varepsilon(z) = \varepsilon_0 + i\varepsilon_1$ and $\bar{\varepsilon}(\bar{z}) = \varepsilon_0 - i\varepsilon_1$. Therefore, $f(z) = z + \varepsilon(z)$ is also holomorphic, which gives rise to an infinitesimal 2D conformal transformation. This implies that the metric tensor also transforms as

$$ds^2 = dzd\bar{z} \rightarrow \frac{df}{dz}\frac{d\bar{f}}{d\bar{z}}dzd\bar{z} \tag{2.21}$$

where $|\frac{df}{dz}|^2$ is the scale factor.

3 The Witt Algebra

Let us assume that $\varepsilon(z)$ is a meromorphic function having isolated singularities outside the open set in a complex plane. Performing a Laurent expansion of $\varepsilon(z)$ around $z = 0$ [10], we get an infinitesimal conformal transformation

$$z' = z + \sum_{n\in\mathbb{Z}} \varepsilon_n \left(-z^{n+1}\right) \quad \text{and} \qquad \bar{z}' = \bar{z} + \sum_{n\in\mathbb{Z}} \bar{\varepsilon}_n \left(-\bar{z}^{n+1}\right) \tag{3.1}$$

The infinitesimal parameters are constant and the number of independent infinitesimal conformal transformations is infinite since $n \in \mathbb{Z}$. The corresponding infinitesimal generators are

$$l_n = -z^{n+1}\partial_z \quad \text{and} \quad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \tag{3.2}$$

The corresponding algebra [Appendix B] is given by

$$\begin{aligned} [l_m, l_n] &= (m-n)l_{m+n} \\ [\bar{l}_m, \bar{l}_n] &= (m-n)\bar{l}_{m+n} \\ [l_m, \bar{l}_n] &= 0 \end{aligned} \tag{3.3}$$

Thus, the algebra of infinitesimal conformal transformations in a 2D Euclidean space is infinite dimensional. Since these commutation relations define two independent algebras of the Witt algebra generated by Eq.(3.2), we treat z and $\bar{z}$ as independent variables decoupled from each other in $\mathbb{C}^2$. In the quantum case, the algebras will be corrected to include an extra term proportional to a central charge (Virasoro algebra). Since the l's and the $\bar{l}$'s commute, the resultant local algebra is the direct sum of the two isomorphic subalgebras. The l's correspond to holomorphic (or chiral) transformations and $\bar{l}$'s correspond to anti-holomorphic (anti-chiral) transformations.

This is a local algebra and not global since the generators are not well-defined on the Riemann sphere $S^2 \simeq \mathbb{C}\cup\infty$, which is the conformal compactification of $\mathbb{R}$. The generators l_n are non-singular at $z=0$ only for $n \geq -1$. To see the behaviour at $z \to \infty$, we perform the change of variables $z = 1/w$ and see that l_n are non-singular as $w \to 0$ only for $n \leq 1$. Therefore, the globally defined conformal transformations on the Riemann sphere are generated by l_{-1}, l_0, l_1.The sub-algebra generated by these corresponds to the global conformal group.

It is clear that l_1 gives raise to translations $z \mapsto z+b$. Since $l_0 = -z\partial_z$, it generates transformations of the form $z \mapsto az$, $a \in \mathbb{C}$. By performing a change of variables $z = re^{i\theta}$, we get

$$l_0 = -\frac{1}{2}r\partial_r + \frac{i}{2}\partial_\theta \quad \text{and} \quad \bar{l}_0 = -\frac{1}{2}r\partial_r - \frac{i}{2}\partial_\theta \tag{3.4}$$

Linear combinations of the two equations gives

$$l_0 + \bar{l}_0 = -r\partial_r \quad \text{and} \quad i(l_0 - \bar{l}_0) = -\partial_\theta \tag{3.5}$$

The first equation corresponds to the generator of 2D dilatations and the second corresponds to the generator of rotations. Similarly, l_1 and $\bar{l}_1$ generates special conformal transformations, which are basically translations for $w = -1/2$. Thus, if $f(z)$ is a mapping which is atmost linear, the only acceptable singularities are the poles since $f(z)$ should not have any branch points and the neighbourhood of an essential singularity should sweep out an entire plane $f(z) = \frac{P(z)}{Q(z)}$. Then, the three operators generates transformations of the form

$$z \to \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{C} \tag{3.6}$$

The same considerations apply for anti-holomorphic transformations.

$$\bar{z} \to \frac{a\bar{z}+b}{c\bar{z}+d} \tag{3.7}$$

For the transformation to be invertible so that $P(z)$ and $Q(z)$ does not have several distinct zeroes, $ad - bc$ must be non-zero and can be normalized to 1. This is the group of projective conformal transformations $SL(2,\mathbb{C})/\mathbb{Z}_2 \approx SO(3,1)$, also known as the Mobius group. The global conformal algebra thus generated is also useful for characterizing properties of physical states. If we work in a basis of eigenstates of the two operators l_0 and $\bar{l}_0$ and denote their eigenvalues by h and $\bar{h}$, respectively, such that h and $\bar{h}$ are independent, real quantities, and not complex conjugates of one another, then h and $\bar{h}$ are known as the conformal weights of the state. Since $l_0 + \bar{l}_0$ and $i(l_0 - \bar{l}_0)$ generate dilatations and rotations, respectively, the scaling dimension Δ and the spin s of the state are given by $\Delta = h + \bar{h}$ and $s = h - \bar{h}$.

4 The Central Charge and Virasoro Algebra

The central extension to the Witt algebra is called the Virasoro algebra. For ordinary quantum field theories with a finite dimensional algebra, the central extension is trivial. But, we have a non-trivial central extension for the infinite dimensional Witt algebra. When it comes to the Witt algebra of infinitesimal conformal transformations, we take into consideration the central extension of a Lie algebra by a central charge. By doing so, projective representations become true representations. On expressing the elements of the central extension of the Witt algebra in terms of $L, \bar{L}$, we get the Virasoro algebra as

$$\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n} + cg(m,n) \\
&[\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n} + \bar{c}g(m,n) \\
&[L_m, \bar{L}_n] = 0
\end{aligned} \tag{4.1}$$

where $c \in \mathbb{C}$ and $g(m,n)$ is anti-symmetric in its arguments i.e. $g(m,n) = -g(m,n)$ so that it is compatible with the anti-symmetry of the Lie bracket. We see that $g(m,n) = \frac{(m^3-m)}{12}\delta_{m,-n}$. This central extension does not affect the infinite sub-algebra of conformal transformations generated by L_1, L_0 and L_{-1}. If this extra term proportional to c was absent, the quantum algebra would be the same as the classical one. Since the commutator of any two elements of the algebra should also be an element of the algebra, this constant term should not be allowed in the algebra. Thus, we see c as an operator that commutes with any element in the algebra. It follows that this operator is a constant (denoted by c) on any representation of the algebra like the operator itself. Such operators are called central charges or conformal anomalies.

We can set $g(1,-1) = 0$ and $g(n,0) = 0$. Thus, the we can redefine the algebra as

$$\begin{aligned}
[L_1, L_{-1}] &= 2L_0 + cg(1,-1) \\
[L_n, L_0] &= nL_n + cg(n,0) \\
[L_n, L_{-n}] &= \frac{c}{12}(n^3-n) + 2nL_0
\end{aligned} \tag{4.2}$$

The Virasoro algebra [Appendix C] is given by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

Due to this central term, the classical symmetry is not preserved in quantum theory. The generators of the Virasoro algebra is not sufficient enough in cases where conformal symmetry is strictly followed. For instance, the simplest string theory, the bosonic string is constructed by d free bosons, where d is the spacetime dimensions, in which case $c = d$. But, there is a ghost contribution (related to the gauge fixing for 2D gravity) of -26. This leads to the well-known concept of string theory having a critical dimension $d = 26$.

5 Primary Fields

Since the generators of the Witt algebras are expressed in terms of z and $\bar{z}$, we perform complexification of the two dimensional Euclidean space $\mathbb{R}^2 \mapsto \mathbb{C}^2$ by introducing $z = x^0 + ix^1$ and $\bar{z} = x^0 - ix^1$ i.e. $\phi(x^0, x^1) \mapsto \phi(z, \bar{z})$ where $x^0, x^1 \in \mathbb{R}^2$ and $z, \bar{z} \in \mathbb{C}^2$. Consider an arbitrary number of dimensions $d = p + q$. The conformal transformation can be defined by the Jacobian

$$\left| \frac{\partial x'}{\partial x} \right| = \frac{1}{\sqrt{detg'_{\mu\nu}}} = \Omega^{-d/2} \tag{5.1}$$

For dilatations and special conformal transformations, the respective Jacobians are

$$\left| \frac{\partial x'}{\partial x} \right| = \lambda^d \quad \text{and} \quad \left| \frac{\partial x'}{\partial x} \right| = \frac{1}{(1 + 2b.x + b^2x^2)^d}$$

Thus, we define a set of fields A_i which is in general infinite and contains the derivatives of all the fields $A_i(x)$. A field that depends only on z is called a chiral or holomorphic field and one that depends only on $\bar{z}$ is called an anti-chiral or anti-holomorphic field. If a field $\phi(z, \bar{z})$ scales under $z \mapsto \lambda z$ as

$$\phi(z, \bar{z}) \mapsto \phi'(z, \bar{z}) = \lambda^h \bar{\lambda}^{\bar{h}} \phi(\lambda z, \bar{\lambda} \bar{z}) \tag{5.2}$$

then it is said to have conformal dimensions $(h, \bar{h})$, where h is the number of z indices in ϕ and $\bar{h}$ is the number of $\bar{z}$ indices in ϕ. If a field undergoes a conformal transformation under $z \mapsto f(z)$ as

$$\phi(z, \bar{z}) \mapsto \phi'(z, \bar{z}) = \left(\frac{\partial f}{\partial z} \right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}} \right)^{\bar{h}} \phi(f(z), \bar{f}(\bar{z})) \tag{5.3}$$

then it is a primary field of conformal dimensions $(h, \bar{h})$. If the above equation holds true for global conformal transformations $f \in SL(2, \mathbb{C})/\mathbb{Z}_2$, then it is called a quasi-primary field. Such fields are also called $SL(2, \mathbb{C})$ primaries. A primary field is automatically a quasi-primary field. For the theory to be covariant under the transformation, the n-point correlation function satisfies

$$\langle \phi_1(x_1)...\phi_n(x_n)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\Delta_1/d} ... \left|\frac{\partial x'}{\partial x}\right|_{x=x_n}^{\Delta_n/d} \tag{5.4}$$

Not all fields turn out to have this transformation property. The fields that do not follow Eq. (5.3) are called secondary fields.

5.1 Infinitesimal Conformal Transformation of Primary Fields

Consider an infinitesimal conformal transformation $f(z) = z + \varepsilon(z)$ where $\varepsilon(z) \ll 1$. Computing upto the first order in $\varepsilon(z)$,

$$\begin{aligned}
\left(\frac{\partial f}{\partial z}\right)^h &= 1 + h\partial_z\varepsilon(z) + \mathcal{O}(\varepsilon^2) \\
\phi(z+\varepsilon(z), \bar{z}) &= \phi(z) + \varepsilon(z)\partial_z\phi(z,\bar{z}) + \mathcal{O}(\varepsilon^2) \\
\left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} &= 1 + \bar{h}\partial_{\bar{z}}\varepsilon(\bar{z}) + \mathcal{O}(\bar{\varepsilon}^2) \\
\phi(z, \bar{z}+\varepsilon(\bar{z})) &= \phi(\bar{z}) + \varepsilon(\bar{z})\partial_{\bar{z}}\phi(z,\bar{z}) + \mathcal{O}(\bar{\varepsilon}^2)
\end{aligned}$$

Substituting these in Eq. (5.3), we get

$$\phi(z,\bar{z}) \mapsto \phi(z,\bar{z}) + (h\partial_z\varepsilon + \varepsilon\partial_z + \bar{h}\partial_{\bar{z}}\bar{\varepsilon} + \bar{\varepsilon}\partial_{\bar{z}})\phi(z,\bar{z})$$

where $\bar{\varepsilon} = \varepsilon(\bar{z})$. Therefore, the transformation of a primary field under an infinitesimal conformal transformation is given by

$$\delta_{\varepsilon,\bar{\varepsilon}}\phi(z,\bar{z}) = (h\partial_z\varepsilon + \varepsilon\partial_z + \bar{h}\partial_{\bar{z}}\bar{\varepsilon} + \bar{\varepsilon}\partial_{\bar{z}})\phi(z,\bar{z}) \tag{5.5}$$

6 The Energy-Momentum Tensor

The energy–momentum tensor or the stress-energy tensor is a symmetric tensor quantity that describes the density and flux of energy and momentum in spacetime, generalizing the stress tensor of Newtonian physics. It can be deduced from the variation of the Lagrangian action with respect to the metric. Therefore, it gives the behaviour of the theory under infinitesimal transformations $g_{\mu\nu} \mapsto g_{\mu\nu} + \delta g_{\mu\nu}$ where $\delta g_{\mu\nu} \ll 1$. Since the algebra is infinite dimensional in 2D, we do not need the explicit form of action to study the theory. All we need the information encoded in the energy-momentum tensor.

Let us consider Noether's theorem which states that for every continuous symmetry in a field theory, there is a current that is conserved i.e.

$$\partial^\mu j_\mu = 0 \tag{6.1}$$

Considering the conformal symmetry $x^\mu \mapsto x^\mu + \varepsilon^\mu(x)$, we define

$$j_\mu = T_{\mu\nu}\varepsilon^\nu \tag{6.2}$$

The local coordinates are generated by charges constructed from the energy-momentum tensor. If this current is preserved, then for a constant ε^μ,

$$\partial^\mu j_\mu = (\partial^\mu T_{\mu\nu})\varepsilon^\nu = 0 \implies \partial^\mu T_{\mu\nu} = 0 \tag{6.3}$$

Therefore, $T_{\mu\nu}$ is divergence-free. This is the condition for translational invariance. For a more general transformation ε^μ,

$$\begin{aligned}
\partial^\mu j_\mu &= (\partial^\mu T_{\mu\nu})\varepsilon^\nu + T_{\mu\nu}(\partial^\mu \varepsilon^\nu) \\
0 &= 0 + T_{\mu\nu}(\partial^\mu \varepsilon^\nu) \\
&= \frac{1}{2} T_{\mu\nu}(\partial^\mu \varepsilon^\nu + \partial^\nu \varepsilon^\mu)
\end{aligned}$$

Substituting Eq. (2.6),

$$\frac{1}{d}T_{\mu\nu}(\partial.\varepsilon) = 0 \implies T^{\mu}_{\mu} = 0 \tag{6.4}$$

Therefore, in a conformal field theory, $T_{\mu\nu}$ is traceless. For a 2D conformal field theory with Euclidean symmetry, consider the transformation

$$T_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial x^{\mu}}\frac{\partial x^{\beta}}{\partial x^{\nu}}T_{\alpha\beta} \tag{6.5}$$

for $x^0 = \frac{1}{2}(z + \bar{z})$ and $x^1 = \frac{1}{2i}(z - \bar{z})$. On computing the components of $T_{\mu\nu}$ [Appendix D], we find that the two non-vanishing components of the energy–momentum tensor are a chiral and an anti-chiral field: $T_{zz}(z, \bar{z}) = T(z)$ and $T_{\bar{z}\bar{z}}(z, \bar{z}) = \bar{T}(\bar{z})$. The metric referred to the complex planes have $g_{zz} = g_{\bar{z}\bar{z}} = 0$ and $g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}$.

7 Radial Quantization and Conserved Charges

Suppose the standard light cone coordinates are $\sigma^0 \pm \sigma^1$ in Minkowski space. These represent the left and right moving massless fields. Their Lorentzian signature can be obtained by a Wick rotation $w, \bar{w} = \sigma^0 \pm i\sigma^1$, which are the holomorphic and the anti-holomorphic fields. In order to eliminate IR divergence, we compactify the space coordinates on a cirlce of radius R given by

$$\sigma^1 = \sigma^1 + 2\pi$$

This is a cylinder in σ^0, σ^1 coordinates, where σ^0 is the radial component and σ^1 is the angular component. The theory thus obtained is defined on a cylinder of infinite length. This is natural from a string theory point of view since the world sheet of a closed string in Euclidean coordinates is a cylinder. Now, on mapping the cylinder to the complex plane

$$z = e^{\sigma^0 + i\sigma^1} = e^{\sigma^0}.e^{i\sigma^1}$$

We see that the time translations $\sigma^0 \mapsto \sigma^0 + a$ are now mapped to complex dilatations $z \mapsto e^a z$ and the space translations $\sigma^1 \mapsto \sigma^1 + b$ are mapped to rotations $z \mapsto e^{ib} z$. Therefore, the dilatation generator on the conformal plane can be regarded as the Hamiltonian of the system and the Hilbert space is built on the surfaces of constant radius. This procedure for defining a quantum theory on the plane is called radial quantization. From Noether's theorem, since the current associated with the conformal symmetry is preserved, there exists a conserved charge given by

$$Q = \int d\sigma^1 j_0 \quad \text{at constant } \sigma^0 \tag{7.1}$$

i.e. over a fixed time-slice. This conserved charge is the generator of symmetry transformations in the field theory i.e. for an operator A

$$\delta A = [Q, A]$$

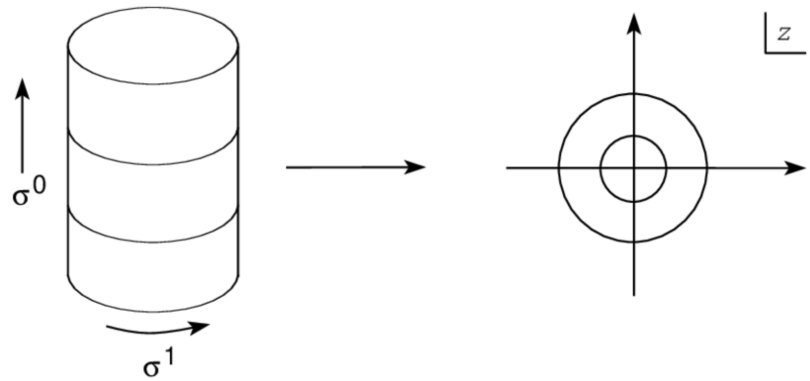

Figure 3: Mapping the cylinder to the complex plane

where the commutator is evaluated at equal times which means that $|z|$ is a constant. This generates the infinitesimal symmetry variations in any field A. Thus, the integral over space $\int d\sigma^1$ becomes the contour integral evaluated anti-clockwise by convention. The conserved charge then becomes

$$Q = \frac{1}{2\pi i} \oint_C (dz T(z)\varepsilon(z) + d\bar{z}\bar{T}(\bar{z})\bar{\varepsilon}(\bar{z})) \tag{7.2}$$

This is the appropriate generalization of Eq. (7.1). In order to specify what other fields lie inside the contour, we find the variation of a field ϕ

$$\delta_{\varepsilon,\bar{\varepsilon}}\phi(w,\bar{w}) = \frac{1}{2\pi i} \oint_C dz[T(z)\varepsilon(z), \phi(w,\bar{w})] + \frac{1}{2\pi i} \oint_C d\bar{z}[\bar{T}(\bar{z})\bar{\varepsilon}(\bar{z}), \phi(w,\bar{w})] \tag{7.3}$$

From quantum field theory, correlation functions are time-ordered products since in such a theory where energy is bounded from below, the Euclidean space Green function converges for operators that are time-ordered. Its analytic continuation gives the desired solution to the Minkowski space equations of motion on the cylinder. The radial ordering operator is defined as

$$R(A(z), B(w)) = \begin{cases} A(z)B(w) & |z| > |w| \\ B(w)A(z) & |z| < |w| \end{cases}$$

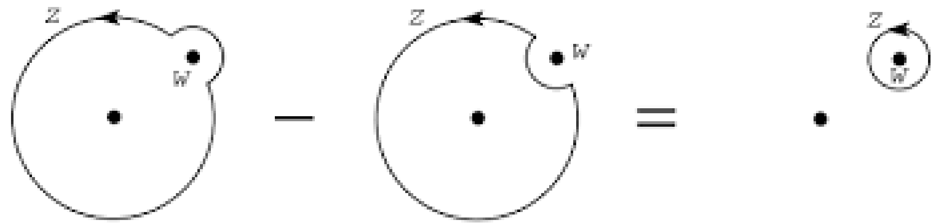

Figure 4: Evaluation of equal-time commutator on the conformal plane

The contour integral is given by

$$\begin{aligned}\oint dz[A(z), B(w)] &= \oint_{|z|>|w|} dz A(z)B(w) - \oint_{|z|<|w|} dz B(w)A(z) \\ &= \oint_{C(w)} dz R\left(A(z)B(w)\right)\end{aligned} \tag{7.4}$$

The equal time commutator of A with the spatial integral of B becomes the contour integral of the radially ordered product of A and B. The sum of the contour integrals is illustrated in Fig. 4 where w is a branch point. Therefore, Eq. (7.3) becomes

$$\delta_{\varepsilon,\bar{\varepsilon}}\phi(w,\bar{w}) = \frac{1}{2\pi i}\oint_{C(w)} dz\,\varepsilon(z) R\left(T(z),\phi(w,\bar{w})\right) + \frac{1}{2\pi i}\oint_{C(\bar{w})} d\bar{z}\,\bar{\varepsilon}(\bar{z}) R\left(\bar{T}(\bar{z}),\phi(w,\bar{w})\right) \tag{7.5}$$

Comparing this result with Eq. (5.5), we get

$$\begin{aligned} h\partial_w\varepsilon(w)\phi(w,\bar{w}) &= \frac{1}{2\pi i}\oint_{C(w)} dz\, h\frac{\varepsilon(z)}{(z-w)^2}\phi(w,\bar{w}) \\ \varepsilon(w)\partial_w\phi(w,\bar{w}) &= \oint_{C(w)} dz\frac{\varepsilon(z)}{z-w}\partial_w\phi(w,\bar{w})\end{aligned} \tag{7.6}$$

This result follows from the Cauchy Integral Formula in complex analysis. This defines the qunatum energy-momentum tensor. For a bi-holomorphic function, the short distance singularities of the chiral and the anti-chiral fields must be given by

$$\begin{aligned} R\left(T(z)\phi(w,\bar{w})\right) &= \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\phi(w,\bar{w}) + ... \\ R\left(\bar{T}(\bar{z})\phi(w,\bar{w})\right) &= \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\phi(w,\bar{w}) + ... \end{aligned} \tag{7.7}$$

This is called the operator product expansion (OPE). It describes the algebraic product structure on the space of quantum fields. The OPE says that two local operators inserted at nearby points can be approximated at one of the points by a string of operators. This operator equation holds as an operation insertion inside a radially-ordered correlation function. OPE's give us the singularities that occur when two operators approach each other. We can also define a field to be primary if the OPE between the energy-momentum tensor and the field follows the said form. From now on, we assume the radial ordering for a product of fields to be written as $A(z)B(w)$ instead of $R\left(A(z)B(w)\right)$ for convenience.

7.1 OPE of the Energy-Momentum Tensor

As we have stated, not all fields satisfy the conformal transformation property given in Eq. (5.3). For example, derivative fields are secondary fields having singularities higher than the double pole singularities in Eq. (7.7). In a conformal field theory, fields can be grouped into families such that each family contains one primary field and an infinite set of secondary or descendant fields. The primary field is considered as the highest weight of the representation. The energy-momentum tensor is an example of a field that does not obey Eq. (5.3). This can be shown from the OPE of the energy-momentum tensor with itself. Let us take the Laurent expansion of the tensor

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n \quad \text{where} \quad L_n = \frac{1}{2\pi i}\oint dz\, z^{n+1}T(z) \tag{7.8}$$

where L_n are the Laurent modes. A similar relation is defined for the anti-chiral part as well. Using this expansion for the conserved charge in Eq. (7.2) and considering the transformation $\varepsilon(z) = -\varepsilon_z z^{n+1}$, we get

$$\begin{aligned} Q_n &= \oint \frac{dz}{2\pi i} T(z)(-\varepsilon_n z^{n+1}) \\ &= -\varepsilon_n \sum_{m\in\mathbb{Z}} \oint \frac{dz}{2\pi i} L_m z^{n-m-1} \\ &= -\varepsilon_n L_n \end{aligned}$$

We have already calculated the commutator of Virasoro algebra. On performing integration by parts to evaluate $\partial_w T(w)$, we obtain the correct form of OPE between two energy-momentum tensors as

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + ... \tag{7.9}$$

where $|z| > |w|$ and c is the central charge whose value depends on the theory under consideration. Since $\langle T(z)T(0)\rangle = \frac{c/2}{z^4}$, we can expect that $c \geq 0$ in a theory with positive semi-definite Hilbert space. Similar considerations apply for $\bar{T}$ as well. The energy-momentum tensor is thus a field which is quasi-primary of conformal dimensions (2,0), but not Virasoro primary. However, it can be a primary field if the central charge vanishes.

8 Conformal Ward Identities

Ward identities are the quantum manifestations of symmetries. They are identities satisfied by the correlation functions as a reflection of the symmetries of a theory. One can derive them by considering the behaviour of n-point functions under a conformal transformation in some localized region containing all the operators. Let us consider a collection of operators located at w_i inside a contour z. In order to perform a conformal transformation, we integrate $\varepsilon(z)T(z)$ around the contour. By analyticity, this contour can now be deformed to a sum of terms such that each term comes from a contour around each operator w_i as shown in Figure 5.

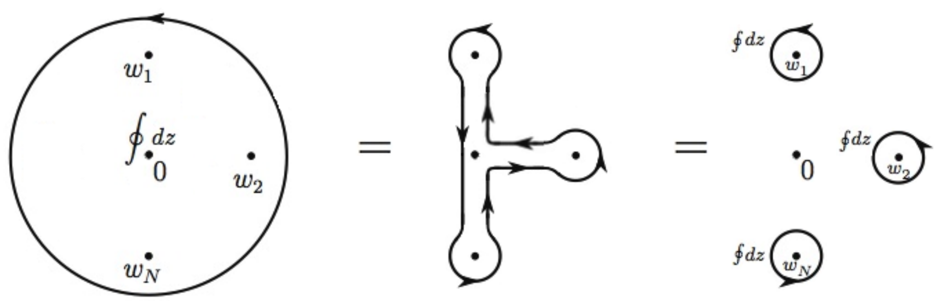

Figure 5: Deformation of contour integrals

For the primary fields ϕ_i,

$$
\begin{aligned}
\left\langle \oint \frac{dz}{2\pi i}\varepsilon(z)T(z)\phi_1(w_1,\bar{w}_1)...\phi_n(w_n,\bar{w}_n) \right\rangle &= \sum_{i=1}^{n}\langle\phi_1(w_1,\bar{w}_1)...\left(\oint_{C(w_i)}\frac{dz}{2\pi i}\varepsilon(z)T(z)\phi_i(w_i,\bar{w}_i)\right) \\
&\quad ...\phi_n(w_n,\bar{w_n})\rangle \\
&= \sum_{i=1}^{n}\langle\phi_1(w_1,\bar{w}_1)...\left(h_i\partial\varepsilon(w_i)+\varepsilon(w_i)\partial_{w_i}\right)\phi_i(w_i,\bar{w}_i) \\
&\quad ...\phi_n(w_n,\bar{w_n})\rangle
\end{aligned}
$$

We have used Eq. (7.6) in the last line. We can use Eq. (7.7) to rewrite the equation as

$$\begin{aligned} 0 = \oint_{C(w_i)} \frac{dz}{2\pi i} \varepsilon(z) [\langle T(z)\phi_1(w_1,\bar{w}_1)...\phi_n(w_n,\bar{w}_n)\rangle \\ - \sum_{i=1}^{n} \left(\frac{h_i}{(z-w_i)^2} + \frac{1}{z-w_i}\partial_{w_i} \right) \langle \phi_1(w_1,\bar{w}_1)...\phi_n(w_n,\bar{w}_n)\rangle] \end{aligned} \tag{8.1}$$

Since this holds true for any infinitesimal conformal transformation ε, we omit the integral. Therefore, we get

$$\langle T(z)\phi_1(w_1,\bar{w}_1)...\phi_n(w_n,\bar{w}_n)\rangle = \sum_{i=1}^{n} \left(\frac{h_i}{(z-w_i)^2} + \frac{1}{z-w_i}\partial_{w_i} \right) \langle \phi_1(w_1,\bar{w}_1)...\phi_n(w_n,\bar{w}_n)\rangle \tag{8.2}$$

Thus, the correlation functions are meromorphic functions of z with the presence of singularities at the positions of operators. The residues at these singularities can be computed by the conformal properties of the operators. We can use this result to show that the 4-point correlation functions involving degenerate fields satisfy hypergeometric differential equations.

9 Unitarity and Highest Weight Representation

There are many representations of the Virasoro algebra. We are interested in its unitary highest weight representations. In field theory, unitarity is a statement of conservation of probability. If all the generators L_n are recognized as operators on a Hilbert space such that $L_n^\dagger = L_{-n}$, then the representation is said to be unitary. This follows from the energy-momentum tensor equality of

$$T^\dagger(z) = \sum \frac{L_n^\dagger}{\bar{z}^{n+2}} \quad \text{and} \quad T\left(\frac{1}{\bar{z}}\right)\frac{1}{\bar{z}^4} = \sum \frac{L_n}{\bar{z}^{-n-2}}\frac{1}{\bar{z}^4}$$

i.e. $T(z)$ is Hermitian [10]. Before we go into the details of this, let us define the so-called in- and out- states. In Euclidean field theory, we associate states with operators by the definition given below

$$|A_{in}\rangle = \lim_{\sigma^0 \to -\infty} A(\sigma^0, \sigma^1)|0\rangle = \lim_{\sigma^0 \to -\infty} e^{H\sigma^0} A(\sigma^1)|0\rangle$$

Since $\sigma^0 \to -\infty$ on a cylinder maps to the zero of the complex plane, we can rewrite the in-state as

$$|A_{in}\rangle \equiv \lim_{z,\bar{z}\to 0} A(z,\bar{z})|0\rangle \tag{9.1}$$

The out state is defined when $z \to \infty$. We consider $z = 1/w$ i.e. relating a neighbourhood about ∞ on the Riemann sphere to that about the origin. If $\tilde{A}(w,\bar{w})$ is the operator in the coordinates for which $w \to 0$ corresponds to the point in ∞, then we can define the out-state as

$$\langle A_{out}| \equiv \lim_{w,\bar{w}\to 0} \langle 0|\tilde{A}(w,\bar{w}) \tag{9.2}$$

For a primary field, we get

$$\begin{aligned}\tilde{A}(w,\bar{w}) &= A\left(f(w),\bar{f}(\bar{w})\right)\left(\partial f(w)\right)^h \left(\bar{\partial}\bar{f}(\bar{w})\right)^{\bar{h}} \\ &= A\left(\frac{1}{w},\frac{1}{\bar{w}}\right)(-w^{-2})^h(-\bar{w}^{-2})^{\bar{h}}\end{aligned}$$

From the definition of adjoint for Euclidean-space fields corresponding to Hermitian fields in Minkowski space

$$[A(z, \bar{z})]^\dagger = A\left(\frac{1}{\bar{z}}, \frac{1}{z}\right) \frac{1}{\bar{z}^{2h}} \frac{1}{z^{2\bar{h}}}$$

we get the relation between the in- and out- states as

$$\langle A_{out}| = |A_{in}\rangle^\dagger \tag{9.3}$$

In the complex plane, the in- and out- states play an asymmetric role. The highest weight representation is defined as the representation that contains a state with a smallest value of L_0. It follows from the result that L_n decreases the eigenvalue of L_0 by n

$$L_0 L_n|\psi\rangle = (L_n L_0 - nL_n)|\psi\rangle = (h - n)L_n|\psi\rangle$$

if $L_0|\psi\rangle = h|\psi\rangle$. This implies that if h is the highest state, then it is annihilated by all generators with $n > 0$

$$L_n|h\rangle = 0 \quad \text{for } n > 1 \tag{9.4}$$

If we act L_0 on a highest weight state $|h\rangle$ to get $|h\rangle'$, then the Virasoro algebra tells us that $L_n|h\rangle' = 0$ for $n > 1$. This means that L_0 maps highest weight states to highest weight states. Since L_0 is Hermitian, we diagonalize it on the highest weight state in order to assume $L_0|h\rangle = h|h\rangle$. Meanwhile, the negative modes i.e. $L_n, n < 0$ is used to generate states known as the descendants. From $[L_0, L_n] = -nL_n$, we see that L_0 can have any integer eigenvalue. $[c, L_n] = 0$ implies that the adjoint representation has a central charge of zero. Therefore, it is a unitary representation. Since $L_0 \pm \bar{L_0}$ are the generators of dilatations and rotations, $h \pm \bar{h}$ are said to be the scaling dimension and the Euclidean spin of the state, respectively. The generators L_1, L_{-1} and L_0 and their corresponding anti-chiral generators correspond to the finite dimensional subalgebra of global conformal transformations. $L_1, L_2, ...$ correspond to additional raising operators and $L_{-1}, L_{-2}, ...$ correspond to lowering ones i.e. L_n raises the conformal dimension by n and L_{-n} lowers it by n.

The unitary highest weight representations are described by two real numbers h and c.

On a representation, c has a constant value since all the generators commute with it. L_0 does not have a constant representation. However, we can give a unique definition of h as its highest weight state eigenvalue. The norm of a state can be expressed completely in terms of c and h, which means that zero or negative norm condition depends on these two values. Therefore, two representations that has the same c and h are equivalent Virasoro representations. From Eq. (4.2), we have

$$\|L_{-n}|0\rangle\| = \langle 0|L_{-n}^{\dagger}L_{-n}|0\rangle = \langle 0|L_n L_{-n}|0\rangle = \frac{c}{12}(n^3 - n) \tag{9.5}$$

where $|0\rangle$ is the vacuum state. It can be called the $SL(2,\mathbb{C})$ invariant vacuum since it is annihilated by both $L_{\pm 1,0}$ and $\bar{L}_{\pm 1,0}$. Since in a Hilbert space, all states must have non-negative norms, for $n \geq 2$ implies $c \geq 0$. If norm is zero, then the state vanishes. Similarly, we can compute the same for any highest weight state

$$\|L_{-n}|h\rangle\| = \langle h|L_n L_{-n}|h\rangle = \left(\frac{c}{12}(n^3 - n) + 2nh\right)\langle h|h\rangle \tag{9.6}$$

If the norm of the highest weight state does not vanish, then $c \geq 0$. If $c = 0$ in a unitary representation, it gives only one state (the vacuum state), which is trivial. For a large n, we must have $c > 0$ and for $n = 1$, we must have $h \geq 0$.

10 Descendant Fields

The representations of the Virasoro algebra starts with one primary field. The remaining fields are obtained by its successive OPE's with the energy-momentum tensor i.e. by commuting L_{-n}'s with the primary fields. The descendant fields give descendant states when acted on the vacuum state. They can be given by

$$T(z)\phi(w,\bar{w}) = \sum_{n\geq 0}(z-w)^{n-2}\phi^{-n}(w,\bar{w}) \tag{10.1}$$

Let us take one term

$$\phi^{-n}(w,\bar{w}) = \oint \frac{dz}{2\pi i}\frac{1}{(z-w)^{n-1}}T(z)\phi(w,\bar{w})$$

We know that

$$\phi^{-n}(0,0)|0\rangle = \oint \frac{dz}{2\pi i}\frac{1}{z^{n-1}}T(z)\phi(0,0)|0\rangle = L_{-n}\phi(0,0)|0\rangle \tag{10.2}$$

Thus, ϕ^{-n} generated the L_{-n} descendants of $|h,\bar{h}\rangle$. The conformal weight of the field is given by $(h+n,\bar{h})$. Energy-momentum tensor is a simple example of a descendant field.

$$L_{-2}1(w) = \oint \frac{dz}{2\pi i}\frac{1}{z-w}T(z)1 = T(w) \implies 1^{-2}(w) = (L_{-2}1)(w) = T(w) \tag{10.3}$$

i.e. it is a level 2 descendant of the identity operator. This is why the OPE of the energy-momentum tensor with itself is not a primary field. Thus, the information required to completely specify a 2D conformal field theory consists of the conformal weights of the Virasoro highest weight states and the operator product expansion three-point function constants between the relevant primary fields.

11 Conformal Bootstrap Method

The conformal bootstrap method gives a powerful way to constrain the conformal weights and three-point function constants. To get a consistent theory, there must be some consistency conditions imposed on the CFT data i.e. we impose global constraints on the correlation functions. Let us take a scalar 4-point function

$$\langle \phi_i(z_1, \bar{z}_1)\phi_j(z_2, \bar{z}_2)\phi_l(z_3, \bar{z}_3)\phi_m(z_4, \bar{z}_4)\rangle$$

We surround two of the operators, say ϕ_i and ϕ_j, by a sphere and expand into the radial quantization states on this sphere. Taking $z_1 \mapsto z_2, z_3 \mapsto z_4$ and finding its schematic result, we get something called the conformal partial wave expansion as depicted in Figure 6. We can also calculate the 4-point function by taking the operators ϕ_i and ϕ_l. The

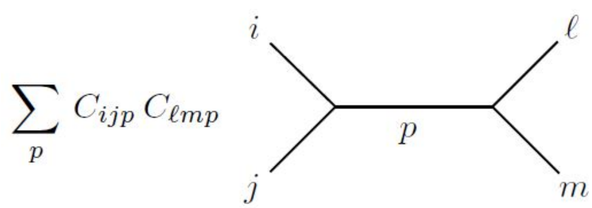

Figure 6: Schematic representation of conformal partial wave expansion

associativity of the algebra says that these two methods yield the same result. This is a necessary consistency requirement known as the crossing symmetry. This condition is also known as the conformal bootstrap condition or the OPE associativity. The procedure of solving these equations to get the conformal field theory is known as the conformal bootstrap method. Fig. 6 represents the duality of the 4-point function. But, crossing symmetry of five or higher point functions do not give additional constraints. In 2D, we consider only the primary fields. The unitary conditions are almost the same as that of the higher dimensions when $c \geq 1$. But, when $0 < c < 1$, the unitarity is restrictive i.e.

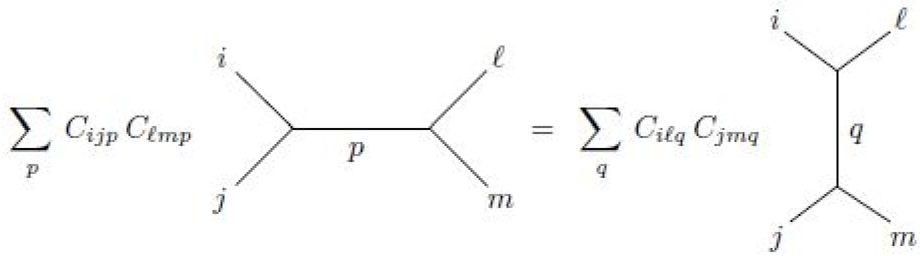

Figure 7: Crossing symmetry

only discrete values of c are allowed

$$c = 1 - \frac{6}{m(m+1)} \tag{11.1}$$

where $m = 3, 4, \ldots.$ Such models are called minimal models. This approach reduces the problem to a finite dimensional linear algebra problem since it has finite number of primaries and all the operator dimensions are known to us. The simplest minimal model is where $c = 1/2$. This model corresponds to the 2D Ising model at the critical temperature.

12 The Kac Determinant

Every sensible representation of the Virasoro algebra is characterized by highest weight states. First, let us discuss some properties of Hilbert space of a CFT. We can consider L_n's, $n > 0$ as an infinite collection of harmonic oscillator annihilation operators and $L_n^\dagger$'s $= L_{-n}$'s as creation operators. In Hilbert space, every state can be expressed as a linear combination of primary and descendant states. At a given excitation level i.e. eigenvalues of L_0 being $n+h$, there are descendant fields that can be expressed as linear combinations of the states

$$L_{-n_1}...L_{-n_k}|h\rangle, \quad \sum_i n_i = n \tag{12.1}$$

According to the normal ordering of products, which says that the creation operators are to the left, the generators are assumed to be ordered $n_i > n_j$ if $i < j$. This collection of states in Eq. (12.1) is called the Verma module of $|h\rangle$. This set is closed under the action of any Virasoro generator. If h is a highest weight state, we can build the set of states as

level	dimension	state
0	h	$\|h\rangle$
1	$h+1$	$L_{-1}\|h\rangle$
2	$h+2$	$L_{-2}\|h\rangle, L_{-1}^2\|h\rangle$
3	$h+3$	$L_{-3}\|h\rangle, L_{-1}L_{-2}\|h\rangle, L_{-1}^3\|h\rangle$
	...	
N	$h+N$	$P(N)$ states

which is the Verma module. Null state is a linear combination of states that vanishes. All the null states and their descendants are removed from the Verma module when the representation of Virasoro algebra with highest weight is constructed. Considering the linear combinations of states that vanishes, at level 1, the only possibility is $L_{-1}|h\rangle = 0$ which implies $h = 0$ or $|h\rangle = |0\rangle$. At level 2, $L_{-2}|h\rangle + aL_{-1}^2|h\rangle$ may be zero for some a. This is not enough to check whether each of these states separately has a positive norm due to the linear combinations resulting in zero or negative norm. To deal with this, we define a matrix

$$
\begin{aligned}
K_2 &= \begin{pmatrix} \langle h|L^{\dagger}_{-2}L_{-2}|h\rangle & \langle h|L^{\dagger}_{-2}L_{-1}L_{-1}|h\rangle \\ \langle h|(L_{-1}L_{-1})^{\dagger}L_{-2}|h\rangle & \langle h|(L_{-1}L_{-1})^{\dagger}L_{-1}L_{-1}|h\rangle \end{pmatrix} \\
&= \begin{pmatrix} 4h + c/2 & 6h \\ 6h & 4h(1+2h) \end{pmatrix}
\end{aligned}
\tag{12.2}
$$

It is a Hermitian matrix i.e. it has real eigenvalues. If the determinant is negative or zero, there will be an eigenvalue which is negative or zero. A zero eigenvector of this matrix gives a linear combination that results in zero norm that should vanish in a positive-definite Hilbert space. A null state for some h occurring at level n implies that the determinant at some level N has $[P(N-n)]^{th}$ order zero. At the N^{th} level, the analogous matrix is K_N and is of the form

$$\langle h|L_{m_l}...L_{m_1}L_{-n_1}...L_{-n_k}|h\rangle$$

where $\sum m_i = \sum n_j = N$. If the determinant is zero, the norm of linear combinations is zero for that c, h values. If it is negative, then there is an odd number of negative eigenvalues and the representation has states of negative norm and hence is not a unitary representation of the Virasoro algebra. This determinant is called the Kac determinant and is given by

$$det\,(K_N(c,h)) = \alpha_N \prod_{pq \leq N} (h - h_{p,q}(c))^{P(N-pq)} \tag{12.3}$$

where α_N is a constant, which is independent of c and h. We can study the behaviour of the Kac determinant starting from an asymptotic value, where we know that all the eigenvalues are positive. Reparametrizing c in terms of m

$$m = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{25-c}{1-c}} \tag{12.4}$$

This allows us to write $h_{p,q}$'s as

$$h_{p,q}(m) = \frac{[(m+1)p - mp]^2 - 1}{4m(m+1)} \tag{12.5}$$

These values have the symmetry where $p \mapsto m - p, q \mapsto m + 1 - q$. From convention, we choose the branch $m \in (0, \infty)$ for $c < 1$.

Let us determine the order of the Kac determinant. This can be computed by noting that the highest power of h in the determinant comes from the product of the diagonal elements of the matrix i.e. the elements that gives maximum L_0's generated by commuting L_k's with L_{-k}'s. For a state $L_{-n_1}...L_{-n_k}|h\rangle$, the diagonal elements contribute to a quantity proportional to h^k. Therefore, the order is given by

$$\nu_N = \sum_N k = \sum_{pq \leq N} P(N - pq) \tag{12.5}$$

where $\sum_{i=1}^{k} n_i = N$. The order of the polynomial on the RHS is the same as that of the determinant implying the states have exhausted all the zeroes and the determinant can be determined up to a constant.

13 Statistical Mechanical Systems

Let us discuss about the unitary representations of Virasoro algebra. Statistical mechanical systems that are described near a second order phase transition is expected to be described by a unitary theory. As we have stated already, if the Kac determinant is negative, the representation is not unitary. As far as a positive or a zero determinant is concerned, from Eq. (12.4), for $c > 1, h \geq 0$, there are no zeroes in the Kac determinant. For $1 < c < 25$, m is not real and hence from Eq. (12.5), $h_{p,q}$'s are either imaginary or negative. For $c \geq 25$, we choose the branch $-1 < m < 0$ and see that all $h_{p,q}$'s are negative. Now, if the determinant is non-vanishing in this region, all the eigenvalues are positive since the matrix will be dominated by the diagonal elements when h is large. Therefore, we can have unitary representations since the determinant never vanishes for $c \geq 1$, $h \geq 0$, which implies that all eigenvalues must be positive in the entire region. At the boundary $c = 1$, the determinant vanishes, but does not become zero. For the region $0 < c < 1$, $h > 0$, we draw the vanishing curves $h = h_{p,q}(c)$ in the h-c plane by rewriting Eq. (12.5) in terms of

$$h_{p,q}(c) = \frac{1-c}{96}\left[\left((p+q) \pm (p-q)\sqrt{\frac{25-c}{1-c}}\right)^2 - 4\right] \tag{13.1}$$

These set of curves are given in Figure 7. We can see that a point in the region $0 < c < 1$, $h > 0$ can be connected to the region $c > 1$ by a path that crosses a vanishing curve of the Kac determinant. The eigenvalue that crosses through zero is the one causing the vanishing in the first place. The determinant reverses sign when it passes through the vanishing curve and there should be a negative norm at that level. Further analysis reveals that there is an extra negative norm state everywhere on the vanishing curves except where they intersect. The unitary representations of Virasoro algebra are not excluded in this discrete set of points. They occur at values of the central charge given by Eq. (11.1). As we know, $m = 2$ is trivial since $c = 0$. It is also clear that the h, c values occur at infinite intersection points. The first such intersections are at level 3 and 4 for $c = 1/2$, and $h = 1/16$ and $h = 1/2$.

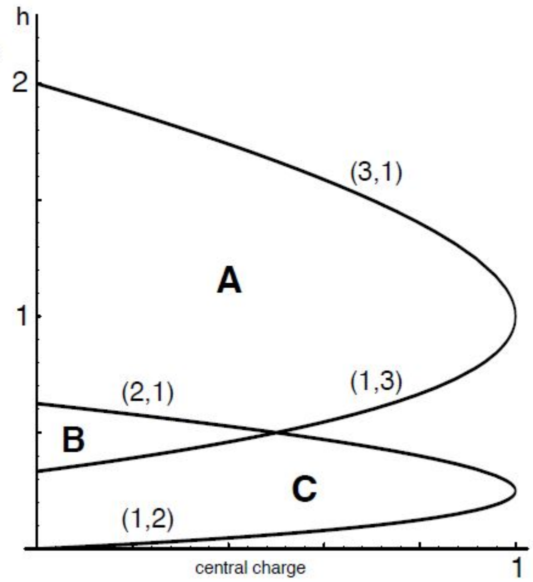

Figure 8: Vanishing curves in h-c plane

13.1 Critical Statistical Models

The Virasoro representation theory allows us to determine the critical indices of 2D systems at second order phase transitions. $m = 3$ in Eq. (11.1) gives $c = 1/2$, which is associated with the Ising model, respectively. Other values of $m = 4, 5, 6$ are associated with the tricritical Ising model, 3-state Potts model and tricritical 3-state Potts model. There may be more than one model at a given $c < 1$. Each model has a second order phase transition at its self-dual point. For a critical statistical mechanical model, the relevant Virasoro representations are usually given by the 2-point correlation functions, which allows us to determine the scaling weights of the operators.

13.2 Critical Ising Model $(m = 3)$

The simplest non-trivial unitary model describes the critical Ising model. The physical systems that we are considering has a non-trivial energy-momentum tensor $\bar{T}(\bar{z})$ with a central charge $\bar{c} = c$. Therefore, the primary fields are described by h and $\bar{h}$, which are the scaling weights. The $(m-1) \times m$ conformal grid with columns as p and rows as q is given below

$\frac{1}{2}$	0
$\frac{1}{16}$	$\frac{1}{16}$
0	$\frac{1}{2}$

As we know, $h_{p,q}$ are left invariant under $p \mapsto m-p$, $q \mapsto m+1-q$ i.e. under a rotation of π. There are a total of $m(m-1)$ values of $h_{p,q}$ if we extend the range of q to $1 \leq q \leq m$. Note that otherwise it is $m(m-1)/2$. We can eliminate the double-counting by restricting to $p+q$ even operators. Let us consider the case of $m = 3$ and $c = 1/2$. The simplest possibility is considering the left and right symmetric fields $\Phi_{p,q}(z, \bar{z}) = \phi_{p,q}(z)\bar{\phi}_{p,q}(\bar{z})$ with conformal weights $h, \bar{h}$:

$$\Phi_{1,1} = (0, 0) \qquad \Phi_{\frac{1}{2},\frac{1}{2}} = \left(\frac{1}{2}, \frac{1}{2}\right) \qquad \Phi_{1,2} = \left(\frac{1}{16}, \frac{1}{16}\right) \tag{13.2}$$

where the $(0, 0)$ field is the identity operator.

13.3 Critical Exponents

Thermodynamic quantities scale as we approach the critical point. This behaviour is defined by critical exponents that are related to the scaling dimensions. As we approach the critical point i.e. as the temperature increases to the Curie temperature T_c, thermal fluctuations become large and a ferromagnetic material becomes paramagnetic. The correlation length $\zeta \to \infty$. The two local scaling operators in the critical Ising model are the Ising spin σ and the energy density ε. They are the continuum versions of lattice spin σ_i and interaction energy $\sigma_i \sigma_{i+1}$. In the high temperature phase, the 2-point function of the order parameter falls off exponentially $\sim exp(-|n|/\zeta)$, where ζ is the correlation length and it depends on temperature. As stated earlier, the correlation length diverges at the critical point and the corresponding mass scale decreases until we have a massless, scale-invariant theory. Therefore, the 2-point correlation function falls off as

$$\langle \sigma_n \sigma_0 \rangle \sim \frac{1}{|n|^{d-2+\eta}} \tag{13.3}$$

where d is the dimension of the system and η is a critical exponent. Similarly, considering the 4-point function criticality, we get another critical exponent ν from the energy operator

$$\langle \varepsilon_n \varepsilon_0 \rangle = \langle \sigma_n \sigma_{n+1} \sigma_0 \sigma_1 \rangle \sim \frac{1}{|n|^{2(d-\frac{1}{v})}} \tag{13.4}$$

Now, from Eq. (1.1), the 2-point function behaves as

$$\langle \sigma_n \sigma_0 \rangle \sim \frac{1}{r^{2\Delta_\sigma}} \tag{13.5}$$

where r is the distance and r dependence is appropriate for scaling dimension $\Delta_\sigma = h_\sigma + \bar{h}_\sigma$ and spin $s_\sigma = h_\sigma - \bar{h}_\sigma = 0$. Therefore, $\Delta_\sigma = 1/8$ and hence the $\left(\frac{1}{16}, \frac{1}{16}\right)$ field in Eq. (13.2) is identified with the spin of the Ising model. The energy operator satisfies

$$\langle \varepsilon_n \varepsilon_0 \rangle \sim \frac{1}{|n|^{2\Delta_\varepsilon}} \tag{13.6}$$

Its scaling weight is $1 = \Delta_\varepsilon = h_\varepsilon + \bar{h}_\varepsilon$. Therefore, $\left(\frac{1}{2}, \frac{1}{2}\right)$ field in Eq. (13.2) is identified with the energy operator of the Ising model. Since $d = 2$, comparing Eq. (13.3) and Eq. (13.4) with Eq. (13.5) and Eq. (13.6), we get

$$\nu = 1 \quad \text{and} \quad \eta = \frac{1}{4} \tag{13.7}$$

These are the independent exponents. The specific heat and the magnetic susceptibility also diverge to infinity. The other exponents are defined as

$$
\begin{aligned}
C &\propto (T - T_c)^{-\alpha} \\
M &\propto (T - T_c)^{\beta} \\
\chi &\propto (T - T_c)^{-\gamma} \\
M &\propto h^{1/\delta}
\end{aligned}
$$

where C is the specific heat, M is the magnetization and χ is the magnetic susceptibility. From the scaling and hyper-scaling laws, in addition to the universality of critical exponents, we obtain the relations between the exponents as

$$
\begin{aligned}
\alpha + 2\beta + \gamma &= 2 \\
\beta\delta &= \beta + \gamma \\
d\nu &= 2 - \alpha \\
2 - \eta &= d\frac{\delta - 1}{\delta + 1}
\end{aligned}
$$

Solving these equations, we obtain the other critical exponents $\alpha = 0, \beta = 1/8, \gamma = 7/4, \delta = 15$.

Therefore, the 2D Ising model has critical exponents given by

$$
(\alpha, \beta, \gamma, \delta, \nu, \eta) = \left(0, \frac{1}{8}, \frac{7}{4}, 15, 1, \frac{1}{2}\right) \tag{13.8}
$$

Part II

4th Semester Work

14 Introduction

In modern physics, all the best theories are written in the framework of quantum field theories. However, this excludes gravity. Gravity is classically described by Einstein's theory of relativity. But, in regions with very large gravity, the spacetime curves so sharp that it may even rip the fabric of spacetime as it does at the singularity of a black hole. In such a situation, Einstein's equations are no longer applicable and a quantum theory of gravity is required to explain what lies behind the horizons of the black hole. Physicists are still trying to formulate this quantum theory of gravity. Quantum field theories give us a hope of finding this theory by unifying gravity with quantum mechanics.

14.1 Geometry of Spacetime

Consider two events in spacetime (ct_1, x_1) and (ct_2, x_2). If the events are separated in space by $\Delta x = x_1 - x_2$ and in time by $\Delta t = t_1 - t_2$, then the invariant interval, which is same in all frames, is the distance between the two points and is given by

$$\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2$$

Including all the spatial axes,

$$\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 \tag{14.1}$$

This interval is invariant under Lorentz transformations, which include rotations and boosts in spacetime. The group of all Lorentz transformations in Minkowski or flat spacetime is the Lorentz group. The coordinates of an event can be written in terms of a four-vector x^μ representing time ($\mu = 0$) and spatial coordinates ($\mu = 1, 2, 3$). The invariant interval can thus be written as an inner product

$$x.x = x^\mu \eta_{\mu\nu} x^\nu \tag{14.2}$$

where $\eta_{\mu\nu}$ is the matrix specified by $diag(-1, 1, 1, 1)$. This is the Minkowski metric defined for flat spacetime.

Einstein's theory of general relativity is the theory of spacetime and gravity. It incorporates the curvature of spacetime. In a curved spacetime containing a gravitational field, the geometry is given by the metric tensor $g_{\mu\nu}$. According to Einstein's equivalence principle, gravity is a manifestation of the curvature of spacetime acting on matter. Unlike other fields that propagate through spacetime, gravitational field is the very metric tensor that describes the curvature of spacetime. Einstein's field equations govern the response of the geometry of spacetime to the distribution of matter in spacetime. It is given by

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{14.3}$$

where G is the Newton's constant of gravitation, $R_{\mu\nu}$ is the Ricci curvature tensor, R is the Ricci scalar and Λ is the cosmological constant [6]. This equation relates the local spacetime curvature to the energy and momentum within that spacetime. Since space is both homogeneous and isotropic, it is maximally symmetric. The maximally symmetric solution to Einstein's equation with a negative cosmological constant is called anti-de Sitter spacetime (AdS).

14.2 Why AdS/CFT?

The efforts directed towards discovering a quantum theory of gravity have led us to string theory and the AdS/CFT correspondence. In 1997, Juan Maldacena conjectured AdS/CFT correspondence, which relates string theory to quantum field theory. Maldacena has proposed that supergravity in an asymptotically AdS spacetime corresponds to a conformal field theory on the boundary at infinity of this spacetime in the large-N limit. It is an explicit manifestation of a more general and underlying holographic principle. What holography does is to map hard problems to simpler ones by a change of dimensions in a completely different background. The AdS/CFT correspondence also provides an insight into the black hole entropy in the context of string theory. Moreover, with the AdS/CFT correspondence, we can calculate very complicated quantities in strongly coupled quantum field theories by computing the corresponding quantities in the dual classical gravitational theory. Duality simply means the equivalence between two seemingly different theories. 2D Ising model is an example of a self-dual model.

	2D Ising Model	AdS/CFT
two sides of the duality	same theory	different theories
degrees of freedom	same in both	different in both
no. of dimensions	same in both	different in both
coupling	strong-weak	strong-weak
solvability	both sides	only one side

Table 1: Dualities in 2D Ising Model and AdS/CFT

15 Classical Field Theory

From special to general relativity, the metric changes to a dynamic tensor field. Thus, GR is an example of a classical field theory. In classical mechanics, the equation of motion for a single particle in one dimension, with coordinate $q(t)$, is given by the principle of least action which states that the path of the particle between two points must be the one corresponding to the least possible value of action (S) given by

$$S = \int dt L(q, \dot{q}) \tag{15.1}$$

where $L(q.\dot{q}) = K - V$ is the Lagrangian, and K and V are the kinetic and potential energies of a point-particle, respectively. The trajectories $q(t)$ for which the action remains stationary for small variations obeys the Euler-Lagrange equation

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial(\dot{q})}\right) = 0 \tag{15.2}$$

For a field theory, we replace the single coordinate $q(t)$ with a set of fields $\Phi^i(x^\mu)$ that are dependent on spacetime. Each field is labelled by i and the action is thus a functional of fields. The Lagrangian can be written as an integral over space of a Lagrange density $\mathcal{L}$ which is given by

$$L = \int d^3x \, \mathcal{L}(\Phi^i, \partial_\mu \Phi^i)$$

Therefore, the action becomes

$$S = \int d^4x \, \mathcal{L}(\Phi^i, \partial_\mu \Phi^i) \tag{15.3}$$

The quantity $\mathcal{L}$ is a Lorentz scalar i.e. it is an invariant scalar under a Lorentz transformation. The final equation of motion, known as the Euler-Lagrange equation for the field theory in flat spacetime [6] is given by

$$\frac{\delta S}{\delta \Phi^i} = \frac{\partial \mathcal{L}}{\partial \Phi^i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)}\right) = 0 \tag{15.4}$$

In order to obtain a Lagrangian appropriate for general relativity, we generalize this expression to a curved, d-dimensional spacetime with a metric having Lorentzian signature. The action becomes

$$S = \int \mathcal{L}(\Phi^i, \partial_\mu \Phi^i) d^d x \tag{15.5}$$

Since $d^d x$ is a density, $\mathcal{L}$ is also a density as their product is a well-defined tensor. Thus, we can write $\mathcal{L}$ as $\sqrt{-g}\mathcal{L}$.

16 Maximally Symmetric Universe

If we define a diffeomorphism from a manifold M to another manifold N as $\phi : M \to N$, then there exists a linear map associated with it, from the space of 1-forms on N to the space of 1-forms on M, called the pullback (ϕ^*). It can pull back any covariant tensor field on N to M. It is said to be a symmetry of a tensor T if the tensor remains invariant after a pullback under ϕ.

$$\phi^* T = T \tag{16.1}$$

In other words, the Lie derivative along a vector field $V^\mu(x)$, which generates a one-parameter family of symmetries ϕ_i, is given by

$$L_V T = 0 \tag{16.2}$$

An isometry is a diffeomorphism of a metric given by $\phi^* g_{\mu\nu} = g_{\mu\nu}$. If the generating vector field is $K^\mu(x)$, then it is known as a Killing field and the Killing equation is given by

$$L_K g_{\mu\nu} = \Delta_\mu K_\nu + \Delta_\nu K_\mu = 0 \tag{16.3}$$

In a d-dimensional manifold, if each dimension can rotate into $(d-1)$ other dimensions without counting the same pair of axes twice, then there are a total of $\frac{1}{2}d(d-1)$ independent rotations. Therefore, the number of independent symmetries is given by d (boosts) $+\,\frac{1}{2}d(d-1)$ (rotations) $= \frac{1}{2}d(d+1)$ (generators). Since the number of isometries are the same as the number of independent Killing fields, we say that this manifold with $\frac{1}{2}d(d+1)$ Killing vectors is maximally symmetric. For such a manifold, the curvature is same everywhere for translation-like symmetries (homogeneity implies it is invariant under a group of translations) and is same in every direction for rotation-like symmetries (isotropy implies it is invariant under a group of rotations). Symmetries of the metric are known as isometries. Examples of isometries of the Minkowski space are translations and Lorentz transformations.

The possible maximally symmetric spaces are classified by the curvature scalar or the Ricci scalar R, which is constant everywhere. Since there is no preferred direction in spacetime, the Riemann tensor must also be the same everywhere i.e. at a particular point which is locally flat, the components of the Riemann tensor must not change under a Lorentz transformation. Since the components of the metric tensor is also invariant under a Lorentz transformation, we can express the components of the Riemann tensor at that point as proportional to a tensor constructed from the invariant metric.

$$R_{\rho\sigma\mu\nu} \propto g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu} \tag{16.4}$$

But, in a maximally symmetric space, all points are equal. Therefore, it must be true everywhere. Contracting both sides of Eq. (16.4) twice, we get

$$R_{\rho\sigma\mu\nu} = \frac{R}{d(d-1)}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}) \tag{16.5}$$

The magnitude of R represents the scaling of the space. The maximally symmetric space

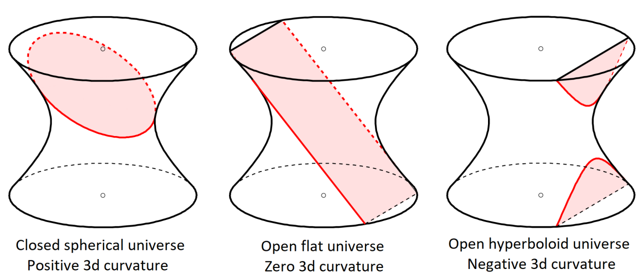

Figure 9: Curvatures of the universe

for $R = 0$ is the flat Minkowski space, while that of $R > 0$ (positive curvature) is the de Sitter space and $R < 0$ (negative curvature) is the anti-de Sitter space. This result follows from the Euler characteristic, a topologically invariant number that describes the shape and structure of the space. In 2D, it is given by $\chi = V - E + F$ where V is the number of vertices, E is the number of edges and F is the number of faces of a given polyhedron embedded in the manifold. It can also be found by integrating the curvature (Gauss-Bonnet theorem)[6].

17 Anti-de Sitter Space

It is a spacetime with a constant negative curvature. This maximally symmetric solution of the Einstein's equation with a negative cosmological constant can be represented as AdS_d and is realized as a hyperboloid embedded in an $(d+1)$ dimensional geometry. Embedding the space of d dimensions allows the properties of the embedded space such as distances and angles to be determined from the simpler properties of the $(d+1)$ dimensional space. The hyperplane is given by

$$X_\mu X^\mu = -l^2 \tag{17.1}$$

where l is the radius of the hyperboloid or the AdS radius. Let us consider AdS_3 in a 4D flat manifold having two time-like directions and whose metric is given by

$$ds^2 = -dx_0^2 - dx_1^2 + dx_2^2 + dx_3^2 \tag{17.2}$$

Note that AdS spacetime itself has only one time-like direction. In order to find the global coordinates of AdS_3, we solve the hyperboloid equation by inducing coordinates (t, ρ, ϕ)

$$\begin{aligned} x_0 &= l\, cosh(\rho)\, cos(t) \\ x_1 &= l\, cosh(\rho)\, sin(\phi) \\ x_2 &= l\, sinh(\rho)\, cos(\phi) \\ x_3 &= l\, cosh(\rho)\, sin(t) \end{aligned}$$

The induced metric on the hyperboloid [6] thus becomes

$$ds^2 = l^2 \left(-cosh^2(\rho)dt^2 + d\rho^2 + sinh^2(\rho)d\phi^2\right) \tag{17.3}$$

These are the global coordinates of AdS_3. An important property of these coordinates is that t is periodic i.e. t and $t + 2\pi$ represent the same point on the hyperboloid. Since ∂_t is time-like everywhere, a curve with constant ρ and ϕ will be a closed time-like curve. We can only have such curves if the causal curves are closed. Thus, we consider the universal covering space of the hyperboloid as $t \in (-\infty, \infty)$ so that there are no closed time-like curves in the covering space. This is the definition of the anti-de Sitter space. For a general AdS$_{d+1}$ space, the metric in global coordinates is given by

$$ds^2 = l^2 \left(-cosh^2(\rho)dt^2 + d\rho^2 + sinh^2(\rho)d\Omega_{d-1}^2\right) \tag{17.4}$$

The conformal diagram or Penrose diagram is derived by performing a coordinate transformation in the radial coordinate ρ

$$cosh(\rho) = \frac{1}{cos(\chi)}$$

Then,

$$ds^2 = \frac{l^2}{cos^2(\chi)} \left(-dt^2 + d\chi^2 + sin^2(\chi)d\rho^2\right) \tag{17.5}$$

We see that the conformal factor has a radial dependence whose range is $0 \leq \chi < \frac{\pi}{2}$. Therefore, AdS space is conformally related to half of Einstein's static universe $(-\frac{\pi}{2}, \frac{\pi}{2})$.

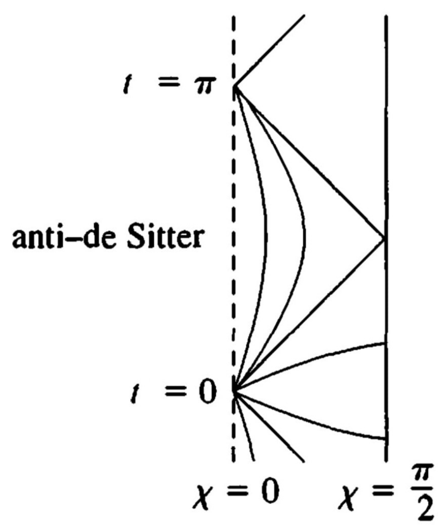

Figure 10: Penrose diagram for AdS space

Figure 10 illustrates a few space-like and time-like geodesics. $\chi = \pi/2$ represents a time-like hypersurface. Since χ only goes to $\pi/2$ and not to π, a space-like slice of this spacetime has the geometry of the interior of a hemisphere of S^3. Thus, it is topologically R^3 and the whole spacetime is R^4. A point on the left side of the diagram represents a point at the spatial origin and one on the right represents S^2 at spatial infinity. Therefore, the $\chi = \pi/2$ hypersurface becomes infinity and since this infinity is time-like, the spacetime is not globally hyperbolic. This hypersurface or conformal boundary is also known as the AdS boundary. The future-pointing time-like geodesics move radially outwards from $t = 0$ and then move inwards to focus on $t = \pi$ and then radially outwards, and so on. This time-like nature of infinity gives way to the AdS/CFT correspondence.

We can say that the Penrose diagram looks like a cylinder whose radial coordinate is ρ, while t and Ω are the surface coordinates [12]. From Figure 10, it is clear that the massless particles reach the boundary at a finite time t, while massive particles do not reach the boundary due to an exponential potential that increases with the radial coordinate.

Considering Poincaré coordinates $(z, \mathbf{x}, t)$, where $\mathbf{x}$ is the collection of x^i's, the AdS metric can be written as

$$ds^2 = \frac{l^2}{z^2}\left(dz^2 - dt^2 + d\mathbf{x}^2\right) \tag{17.6}$$

These coordinates cover a wedge in the global cylinder, where the boundary is at $z = 0$.

18 The Holographic Principle

The holographic principle was first proposed by Gerard 't Hooft and later, a string theory interpretation was given by Leonard Susskind. It states that the description of a volume of space can be thought of as encoded on a lower-dimensional boundary to the region. The two physical theories involved are dual to each other.

18.1 Black Hole Information Paradox

An observer moving with a uniform acceleration through the Minkowski vacuum observes a thermal spectrum of particles. This is the Unruh effect. For a truly thermal spectrum, there must be no correlations between the observed particles. The thermal radiation emitted at one point with frequency ω_1 is observed at another point with frequency

$$\omega_2 = \frac{V_1}{V_2}\omega_1 \tag{18.1}$$

where V is the redshift factor, which is the norm of the Killing vector. Thus, for an observer at the Rindler spatial coordinate $\xi = 0$ moving with a constant acceleration a, the temperature observed is redshifted to $T = ae^{-a\xi}/2\pi$ from that of an observer at $\xi = 0$. As $\xi \to \infty$, the temperature redshifts to zero.

For an observer very near to the event horizon, the spacetime looks essentially flat [6]. Assuming that a freely-falling observer near the black hole observes the quantum state of a scalar field ϕ as the Minkowski vacuum, he will detect the Unruh radiation at a temperature T_1. An observer at infinity detects a thermal radiation near the horizon redshifted to a temperature $T_2 = \frac{V_1}{V_2}T_1$ where $V_2 \to 1$, which gives the surface gravity. But, for an accelerating observer in Schwarzschild space (space surrounding any spherical mass), the Killing vector has a finite norm at infinity. So, the temperature does not redshift all the way to zero, but to some finite value. Therefore, a flux of thermal radiation is emitted from the black hole as seen by the observers far away from the black hole, at a temperature proportional to its surface gravity. This radiation is known as the Hawking radiation. This can be understood by considering vacuum fluctuations near the horizon giving rise to virtual particle-antiparticle pairs spontaneously popping in and out of existence. One of the pair falls into the black hole and the other escapes to infinity. To conserve energy, the one falling in is supposed to have a negative energy, while the other is supposed to have a positive energy. For a far away observer, the black hole seems to have emitted a particle thus reducing its mass. Therefore, Hawking radiation reduces the mass of the black hole ultimately leading to the black hole evaporation. The surface gravity increases with the shrinking of mass, which in turn increases the temperature and more mass is evaporated away. This runaway process proceeds until the entire black hole is evaporated in a finite time.

Any information falling into the black hole is not accessible to an outside observer and is presumed to be hidden behind the event horizon. This is the no-hair theorem, which states that a black hole is only characterized by its mass, angular momentum and charge. If an object collapses into a black hole, the physical properties of the object becomes unmeasurable. However, as the black hole evaporates away, there is no more event horizon that hides the information. Only the Hawking radiation remains. Since these outgoing particles have no hidden correlation, the information is completely lost. This violates unitarity, which implies that quantum information is conserved. This paradox is known as the black hole information paradox.

18.2 Bekenstein-Hawking Formula

The Hawking radiation is emitted by the black hole at a temperature $k_B T = \hbar\kappa/2\pi c$. The second law of thermodynamics requires the black hole to possess an entropy. The no hair theorem is resolved by the area theorem which states that the black hole event horizon will never shrink with time i.e. $dA \geq 0$. This suggests an analogy between the black hole area and entropy. Based on this, Bekenstein stated that black holes carry an entropy proportional to its area. Hawking modified this to what is known now as the Bekenstein-Hawking formula [Appendix E].

$$S_{BH} = \frac{k_B A}{4 l_p^2} \tag{18.2}$$

where l_p is the Planck length and A is the area of the event horizon.

For a given amount of matter, the black hole state is the maximum entropy state [15]. This is the Bekenstein bound. When an object collapses into a black hole, the loss of information is maximum allowed by the laws of physics. This maximum entropy scales with the area of the black hole according to Eq. (18.2). This has motivated the holographic principle. The black hole entropy is proportional to its area, while the statistical entropy is proportional to the volume of the system. We can consider the five dimensional area as the four dimensional volume. This implies that a five dimensional black hole can correspond to a four dimensional statistical system i.e. all the dynamics of a system confined in a volume is encoded in the degrees of freedom living in one dimension less. This is known as the holographic principle and was proposed by Gerard 't Hooft.

An explicit manifestation of holographic principle is the AdS/CFT correspondence. Here, we map the simpler gravitational theory to the complicated quantum field theory. Having a solution on one side helps us understand the other side.

19 The Planar Limit

19.1 Large-N Gauge Theory

Quantum chromodynamics (QCD) is the theory of strong interactions. Since the perturbation theory has only a limited power to solve QCD, an approximate non-perturbative theory called the large-N gauge theory was proposed by Gerard ‘t Hooft. He pointed out that the gauge theories based on the U(N) group simplify as $N \to \infty$ [23]. A U(N) gauge theory has two parameters: gauge theory coupling constant g_{YM} and N colors. For a gauge field A^{μ}, the Lagrangian can be given by

$$\mathcal{L} = \frac{N}{\lambda}\left((\partial A)^2 + A^2 \partial A + A^4\right) \tag{19.1}$$

where we introduce an independent parameter called the ’t Hooft coupling $\lambda = g_{YM}^2 N$. In the Feynman diagram obtained from the Lagrangian, the propagator of the field can be represented by a double-line indicating the flow of a fundamental index. When an index line closes, each index loop gives a factor of N, which enters the Feynman rule thus. Consider a diagram with V vertices, P propagators and F index loops. According to the Feynman rules,

$$\left(\frac{N}{\lambda}\right)^V \left(\frac{\lambda}{N}\right)^E N^F = \lambda^{E-V} N^{V-E+F} \tag{19.2}$$

The diagrams can be summarised as

$$f_0(\lambda)N^2 + f_1()N^0 + f_2(\lambda)N^{-2} + ... \tag{19.3}$$

In the limit of $N \to \infty$, keeping $\lambda = g_{YM}^2 N$ fixed, we see that only the first term of Eq. (19.3) dominates (planar diagrams). This is known as the ’t Hooft limit.

These diagrams can be given a topological meaning by turning them into polyhedra, where each index loop is a facet, each propagator is an edge, and each vertex is also a vertex in the figure. Then, we can apply Euler’s theorem for a polyhedron $\chi = F - P + V = 2 - 2h$,

Figure 11: Vacuum diagrams with double line notation: planar (left) and non-planar (right)

where h is the genus (number of holes) in the polyhedron. Thus, the amplitude of the diagram can be given by

$$g^{2p-2V}N^F = (g^2N)^{F+2h-2}N^{2-2h} \tag{19.4}$$

The topological expansion is obtained by considering $N \to \infty$. From Eq. (19.4), we see that the N dependence is given by the factor N^{2-2h}. Therefore, it is dominated by the diagrams with no holes ($h = 0$) i.e. spheres. The summation of these vacuum diagrams

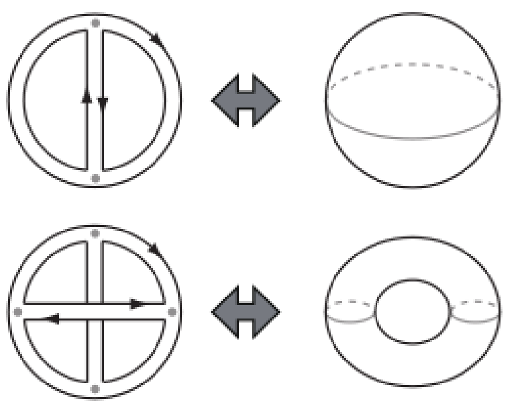

Figure 12: Relationship between diagrams and their topologies: planar diagram corresponds to a sphere (top) and non-planar diagram corresponds to a torus (bottom)

gives the partition function

$$ln(Z_{gauge}) = \sum_{h=0}^{\infty} N^{\chi} f_h(\lambda) \tag{19.5}$$

Thus, we see that the partition function of a large-N gauge theory is a summation over the topologies of 2D surfaces. As the number of propagators increases, the diagram becomes denser and more dominant in the large-N limit. Meanwhile, corresponding topology becomes a smoother 2D surface. Therefore, this limit is easier to evaluate in the topology point of view. Furthermore, the planar two-point function is independent of N while higher point functions are suppressed by $1/N$. This is called the large-N factorization. The stress energy tensor correlation function scales as N^2 in the large-N limit [16].

19.2 String Interactions

String theory is considered as a unified theory including gravity. In this section, we will be looking at the perturbative expansion of string theory [15].

The elementary particles are considered as vibrations of a string. The only fundamental length scale is the string length l_s. There are two types of strings: open (has end-points) and closed (loops) strings. Open strings represent gauge theories. Closed strings represent gravity. An open string can have its end-points on an extended object of the 10D space known as the Dirichlet membrane or D-brane. They are solitons. Thus, there is a localized gauge theory described on the D-brane. An open string can also have its end-points on the same or different D-branes. If there are N coincident D-branes, then they correspond to SU(N) degrees of freedom representing an SU(N) gauge theory.

Since the simplest interaction of an open string is the joining of two strings into one, the end-points of a single string can join to form a closed string which represents a graviton. If there are open strings, there must be closed strings. Therefore, string theory must contain both gauge theory and gravity.

The fundamental coupling constant in string theory is the string coupling constant g_s. It gives the strength of interaction for an emission or absorption of a closed string. A string

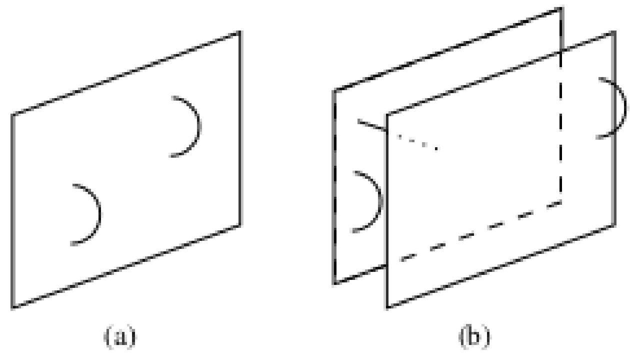

Figure 13: Open strings having their end points on D2-branes (D-brane with 2D spatial extension)

sweeps a 2D worldsheet in spacetime. The worldsheet of a closed string is a cylinder. String interactions are classified by world-sheet topologies. A closed string 1-loop (handle or genus) has an interaction of a virtual closed string proportional to g_s^2. As we add more loops, we get more handles in the diagram, each adding a factor of g_s^2. The vacuum amplitudes have the topology of a sphere (planar diagrams) proportional to $1/g_s^2$. For h loops, the factor is given by g_s^{2h}, where h is related to the Euler characteristic as $\chi = 2-2h$. The amplitudes are then given by $1/g_s^{\chi}$. The Feynman diagram of an interaction is shown

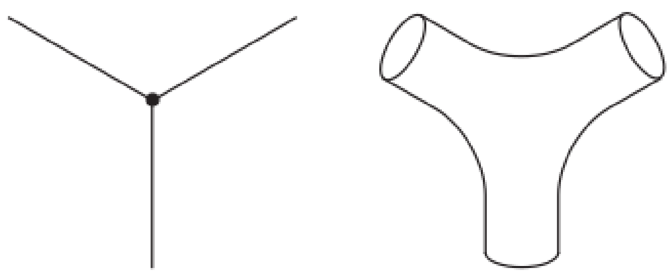

Figure 14: Feynman diagram representing interaction vertex (left) and interaction of a closed string (right)

in Figure 14. The diagram is proportional to g_s since it is the interaction of three external lines.

Dp-branes are D-branes extended in (p+1) dimensions (p spatial dimension and 1 time dimension). The action of Dp-branes [20] is given by

$$S = \frac{1}{(2\pi)^p g_s l_s^{p+1}} \int d^{p+1}x[...] \tag{19.6}$$

We see that it has a $1/g_s$ dependence. So, Dp-branes are non-perturbative objects in string theory. From the above discussion, we obtain the partition function as

$$ln(Z_{string}) = \sum_{h=0}^{\infty} \left(\frac{1}{g_s}\right)^{\chi} f_h(l_s) \tag{19.7}$$

19.3 Classical Gravity Approximation

String theory predicts an infinite number of elementary particles of mass proportional to $1/l_s$. If we neglect l_s, we get finite elementary particles which can be described by the standard field theories. These field theories become classical when only the lowest order topology expansion exists ($g_s << 1$). Such a classical field theory that describes string theory is called supergavity. The fundamental coupling constants in supergravity are Newton's gravitational constant in 10D (G_{10}) and the gauge field coupling constant (g_{YM}). We consider supergravity in 10D since string theory can be quantized consistently only in 10 dimensions. The Einstein-Hilbert action for supergravity is given by

$$S = \frac{1}{16\pi G_{10}} \int d^{10}x\sqrt{-g}R \tag{19.8}$$

where R is the Ricci scalar and g is the metric tensor. The string coupling constant is related to the Newton's constant as

$$G_{10} \propto g_s^2 \tag{19.9}$$

This is obtained by comparing the two-to-two scattering amplitudes in supergravity and string theory [19]. Comparing Eq. (19.6) to the gauge field action, we see that

$$g_{YM}^2 \propto g_s \tag{19.10}$$

G_{10} has a dimension of $[L]^8$ and the gauge field has a dimension of $[L]^{-1}$. Since the (p+1) action is a dimensionless quantity, we write the above equations as

$$G_{10} \simeq g_s^2 l_s^8 \quad \text{and} \quad g_{YM}^2 \simeq g_s l_s^{p-3} \tag{19.11}$$

Thus, we can say that the classical gravity approximation of string theory in the large-N limit represents a large-N gauge theory that does not include gravity. Comparing the partition functions in Eq. (19.5) and Eq. (19.7), we see that

$$Z_{gauge} = Z_{string} \tag{19.12}$$

and $N^2 \propto \frac{1}{g_s^2} \propto \frac{1}{G}$ such that $\lambda \leftrightarrow l_s$. Since string theory includes gravity, we consider at least 5 dimensions for the theory to be consistent. Such a Poincaré invariant metric is the AdS_5 metric.

20 AdS/CFT Correspondence

This correspondence was first proposed by Juan Maldacena in 1997. It is a conjectured relationship between two physical theories. The statement of the AdS/CFT correspondence, also known as the Maldacena conjecture, can be very crudely given as: *gravitational theory in (d+1)-dimensional AdS spacetime = strongly coupled d-dimensional gauge theory*. This duality is holographic since the gravitational theory is described in an extra dimension. The CFT can be thought of as living on the conformal boundary of AdS. The geometry of AdS is given by $\mathrm{AdS}_5 \times \mathrm{S}^5$, but the sphere does not affect the conformal boundary.

A particle on a geodesic has a conserved energy at infinity. As we have seen earlier, for an observer at infinity, everything is infinitely redshifted at the near horizon region. Thus, the near horizon region can be considered as a low energy limit.

20.1 Dictionary of AdS/CFT

The boundary value of the bulk field ϕ_0 acts as the source for the dual boundary (CFT) operator $\mathcal{O}$. The generator for the correlation functions [2] is given by

$$W(\phi_0) = \int d^d x \phi_0 \mathcal{O} \tag{20.1}$$

The partition function can then be written as

$$e^{W(\phi_0(x))} = e^{\int \phi_0(x)\mathcal{O}}|_{CFT} \tag{20.2}$$

But $\phi_0(x) = \phi(x,z)|_{boundary}$. Thus, the above equation becomes the partition function of the gravity theory with boundary conditions. In asymptotically AdS spacetime, the statement can be rewritten as

$$Z_{AdS} = Z_{gauge} \tag{20.3}$$

This is the GKPW dictionary. We can write the relations between the operators in the boundary theory and their bulk duals as:
stress tensor $T^{\mu\nu} \leftrightarrow$ bulk graviton $g^{\mu\nu}$
U(1) current $J^{\mu} \leftrightarrow$ gauge field A^{μ}
scalar operator $\mathcal{O} \leftrightarrow$ bulk scalar ϕ

CFTs are UV complete theories i.e. interaction between particles become weaker at a high energy scale. This implies that the CFTs are well-defined globally. The gauge symmetries in the bulk thus become global symmetries in the boundary. Eq. (20.3) gives a non-perturbative formulation of a UV complete theory of quantum gravity. When the UV cut-off in the field theory is translated to the bulk theory, it become IR cut-off due to the fact that volume/length becomes infinite in AdS space.

The boundary conditions of the massive bulk scalar are given by

$$\phi(z, t, \mathbf{x}) = leading(t, \mathbf{x}) z^{\triangle_+} + ... + subleading(t, \mathbf{x}) z^{\triangle_-} \tag{20.4}$$

where the mass of the field is related to the scaling dimension as

$$m^2 = \triangle(d - \triangle), \quad \triangle = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 l^2} \tag{20.5}$$

where $\triangle$ are the scaling dimensions, m is the mass, d is the number of dimensions and l is the AdS radius [Appendix C]. So, the mass of the field determines the scaling dimension of the operator. A CFT can be constructed by considering Dp-branes as we will see in the next section.

20.2 IIB String Theory and $\mathcal{N} = 4$ Super Yang-Mills Theory

It is the simplest example demonstrating the AdS/CFT correspondence. Maldacena conjectured [14] that in the 't Hooft limit, the duality is given as *IIB string theory in* $AdS_5 \times S^5$ = *Yang-Mills theory in 4D with* $\mathcal{N} = 4$ *supersymmetry.*

Supersymmetry is the generalization of Poincaré invariance. Type IIB string theory is a superstring theory in 10 dimensions with a maximal supersymmetry of $N = 2$ i.e. there are 32 supercharges. It is a chiral theory (left-right asymmetric). $\mathcal{N} = 4$ super Yang-Mill's theory is a maximally supersymmetric 4 dimensional gauge theory. It is an SU(N) gauge field with matter fields required by supersymmetry. Both the theories are scale-invariant. Since we are interested in a 4D gauge theory, we consider D3-branes on which it is described. The gauge theory is represented as the low-energy limit of the dynamics of a stack of N coincident D3-branes in type IIB string theory. This is the S-duality and is represented by the group $SL(2, \mathcal{Z})$.

For a stack of D3-branes, the spatial dimension is 9 and the brane dimension is 3. Thus, there are 6 scalar fields. Furthermore, the directions transverse to the brane are all isotropic. Since they correspond to the scalar fields, isotropy implies that there is a global SO(6) symmetry associated with the fields. This is called R-symmetry and we get the symmetry group of $\mathcal{N} = 4$ SYM theory as a combination of the conformal symmetry and R-symmetry SO(2,4) $\times$ SO(6).

Since D3-branes have energy, they have an associated energy-momentum tensor that curves the spacetime according to general relativity. Considering the mass density of the brane T_3 and the 6 dimensions transverse to the brane, the Newtonian potential can be given by

$$\frac{G_{10}T_3}{r^4} \simeq \frac{g_s N l_s^4}{r^4}\frac{1}{l_s^4} \tag{20.6}$$

where the RHS follows from Eq. (19.11). Thus, we see that $G_{10}T_3 = g_s N l_s^4$, which can be ignored at the decoupling limit $l_s \to 0$ i.e. the effects of gravity and mass can be ignored.

Thus, from D3-branes, we get $\mathcal{N} = 4$ SYM theory and supergravity in 10D flat spacetime.

However, we cannot ignore gravity at the origin. Therefore, there will be a curved spacetime near the origin $r^4 << g_s N l_s^4$ which can be described by supergravity. The 10D metric for black D3-brane (an extended black hole with a planar horizon) is given by

$$ds^2 = f^{-1/2}(-dt^2 + d\mathbf{x}^2) + f^{1/2}(dr^2 + r^2 d\Omega_5^2) \tag{20.7}$$

where $f = 1 + \frac{l^4}{r^4}$ and $l^4 \simeq g_s N l_s^4$. Considering the near horizon limit $r << l$, we get

$$ds^2 = \frac{r^2}{l^2}(-dt^2 + d\mathbf{x}^2) + l^2 \frac{dr^2}{r^2} + l^2 d\Omega_5^2 \tag{20.8}$$

The first two terms of the equation represent AdS$_5$ and the last term represents a circle S^5 of radius l. Therefore, we get the near horizon geometry as AdS$_5\times$ S^5. This implies that the gravity counterpart also has SO(2,4) $\times$ SO(6) symmetry. Therefore, $\mathcal{N} = 4$ SYM corresponds to supergravity on AdS$_5\times$ S^5. The GKPW relation can be written as

$$Z_{\mathcal{N}=4} = Z_{AdS_5 \times S^5} \tag{20.9}$$

On compactifying S^5 (Kaluza-Klein compactification [22]), the resulting 5D gravity theory is called gauged supergravity in AdS$_5$ and comparing the action to the 10D action, we get $G_5 = G_{10}/l^5$. From the dependencies we have seen previously

$$\begin{aligned} l^4 &\simeq g_s N l_s^4 \\ G_{10} &\simeq g_s^2 l_s^8 \\ g_s &\simeq g_{YM}^2 \end{aligned}$$

we obtain the dictionary of AdS/CFT along with the coefficients as

$$N^2 = \frac{\pi}{2}\frac{l^3}{G_5} \quad \text{and} \quad \lambda = \left(\frac{l}{l_s}\right)^4 \tag{20.10}$$

This is a strong/weak duality. When we consider the semiclassical Einstein gravity, the stringy corrections must be suppressed, which means $N >> 1$ and $\lambda >> 1$ and the CFT is strongly coupled. When the CFT is weakly coupled, $l_s >> l$ which means the stringy corrections are not suppressed and we get a theory of higher spin gravity.

20.3 Correlation Functions in AdS/CFT

If the CFT is weakly coupled, then the theory has many operators which corresponds to massless, high-spin states in the gravity side, which is included in the high energy regime of string theory. So, for a low energy effective field theory, the number of low dimension operators must be less, which means the theory is strongly coupled. Furthermore, the weakly coupled gravity theory requires a large number of degrees of freedom since it is given by the stress tensor 2-point function. So, the CFT will have a large degeneracy of states at high energy. Thus, we require a highly confining CFT theory to account for both the conditions.

We can find the correlation function by differentiating Eq. (20.2) with respect to the field. Each differentiation inserts an operator $\mathcal{O}$, sending a closed string state into the bulk. The stress tensor is an operator for a local field theory.

As we have discussed in Section A, the eigenvalue of the dilatation operator is the scaling dimension

$$[D, \mathcal{O}(0)] = -i\triangle\mathcal{O}(0) \tag{20.11}$$

The large-N gauge theories suppress the connected correlation functions as

$$\begin{aligned} \langle \mathcal{O}\mathcal{O} \rangle &\sim 1 \\ \langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O} \rangle &\sim \langle \mathcal{O}\mathcal{O} \rangle \langle \mathcal{O}\mathcal{O} \rangle + \mathcal{O}(1/N^2) \end{aligned}$$

Moreover, gravity has an approximate Fock space. For the weakly coupled scalar field, we can construct particle states by acting with operators. If two single-trace primary operators

$\mathcal{O}_1$ and $\mathcal{O}_2$ have dimensions $\triangle_1$ and $\triangle_2$, then a third multi-trace primary operator can be given by $\triangle_{\mathcal{O}} = \triangle_1 + \triangle_2 + \mathcal{O}(1/N)$. This does not involve large corrections like the ordinary CFTs.

21 Phase Transitions in AdS/CFT

According to the GKPW relation, we can find the transport coefficients of a perturbed system by computing the QFT partition function. However, for a strongly coupled theory, this would be difficult. So, we approach the problem from the partition function of the dual gravity theory.

$$e^{i\int \phi_0 \mathcal{O}} = e^{iS(\phi)|_{\phi_0=0}} \tag{21.1}$$

where S is the on-shell action for the gravitational theory. But, in a linear response theory, we consider an external source while computing the correlation function of the operators. We are interested in two such fields, namely the massive scalar field and the Maxwell field.

We also consider boundary conditions inside the bulk spacetime. For a time independent perturbation, we consider the regularity condition at the horizon. For a time dependent perturbation, we consider the incoming wave boundary condition at the horizon. A perturbation is given in the background geometry such that the effect of the perturbation is small enough to not affect the geometry. Thus, we get a finite energy-momentum tensor of the perturbation.

We look at phase transitions in AdS/CFT since strongly coupled condensed matter systems with a dual gravity description will give us insights into the problem. In this section, we will be discussing about holographic superconductors as an example of second order phase transition in AdS/CFT.

21.1 Superconductivity

Superconductivity is characterized by zero resistivity or infinite conductivity, and Meissner effect, which expels magnetic field from the material below a certain temperature. In 1950, Landau and Ginzburg described superconductivity in terms of a second order phase transition [15],[13]. The order parameter is a complex scalar field since superconductivity is a macroscopic quantum phenomenon, and the system is coupled with a gauge field. Superconductivity is described by the BCS theory which says that due to the motion of an electron, a distortion is created in the lattice, which is mediated by phonons to another electron inducing an attractive interaction between the electrons. Such an electron pair with opposite spin and momenta forms a Cooper pair. Superconductivity occurs due to the condensation of the Cooper pair and the order parameter corresponds to the macroscopic wave function of the Cooper pair.

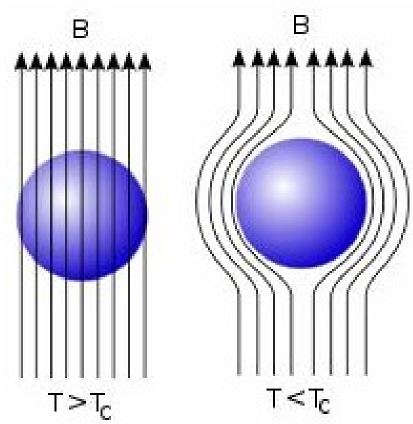

Figure 15: Meissner effect

The two characteristic length scales in superconductivity are the correlation length ξ and the magnetic penetration length λ. The latter specifies the length scale up to which the massive gauge field enters into the superconductor. According to these values, the superconductors are of two types: type I superconductors (exhibits very sharp transition) and type II superconductors (forms vortices where the field enters the material since $\lambda > \xi$).

The linear response function is given by the London equation

$$J_\mu = -\frac{1}{\lambda^2} A_\mu \tag{21.2}$$

from which we conclude that the field decays exponentially inside the superconductor (Meissner effect) and the conductivity diverges.

21.2 Holographic Superconductors

A holographic superconductor is the gravitational dual to a superconductor obtained by coupling AdS gravity to Maxwell field and a charged scalar. Holographic superconductivity means that we can see a 3D image by looking at a 2D superconductor. In the gravity dual, the notion of temperature is given by a black hole in AdS, whose temperature increases with its mass at large radius i.e. it has a positive specific heat. The notion of a non-zero condensate is provided by a static non-zero field outside the black hole. This is called the black hole hair. Thus, a superconductor is described by a black hole that has hair at only low temperatures and not at high temperatures. Such a black hole is the Reissner Nordstrom AdS (RN-AdS) black hole. If the electric field near the horizon is strong enough, then even the maximally charged black hole with no Hawking temperature can create charged particle pairs. But, in AdS space, the charged particles cannot escape to infinity due to the negative cosmological constant and thus the particles settle outside the horizon. These particles give the quantum description of the hair.

The 4D bulk theory is dual to a (2+1)D boundary theory. The gauge symmetries in the bulk corresponds to the global symmetries in the boundary. We consider an Einstein-Maxwell-complex scalar system, the solution to which is the RN-AdS black hole with an additional (energy) dimension of chemical potential. This implies that the temperatures are not equivalent. The Einstein-Maxwell-complex scalar system has an action given by

$$S = \int d^{p+2}x\sqrt{-g}\left(R - 2\Lambda - \frac{1}{4}F_{\mu\nu}^2 - |D_\mu\psi|^2 - m^2|\psi|^2)\right) \tag{21.3}$$

where $\Lambda = -3/l^2$ is the negative cosmological constant coupled to the Maxwell field and the charged scalar with mass $m = -2/l^2$ and charge e, $F_{\mu\nu}$ is the Maxwell's field tensor, $D_\mu = \nabla_\mu - ieA_\mu$ is the dilatation operator and $p = 2$. m becomes sufficiently negative near the horizon to destabilize the scalar field. The last term is the order parameter term. The solution to this system is the RN-AdS black hole with $\psi = 0$. At low temperatures, the solution becomes unstable and the system undergoes a second order phase transition. This is why we assign a non-zero scalar. To analyse the low temperature superconducting phase, we approach it from the familiar high temperature normal phase.

On redefining the matter fields as $\psi \to \psi/e$ and $A_\mu \to A_\mu/e$, the action becomes

$$S = \int d^{p+2}x\sqrt{-g}\left[R - 2\Lambda + \frac{1}{e^2}\left(-\frac{1}{4}F_{\mu\nu}^2 - |D_\mu\psi|^2 - m^2|\psi|^2)\right)\right] \tag{21.4}$$

We see that the backreaction of the matter fields on the metric is suppressed as $e \to \infty$. This is known as the probe limit. Thus, the Maxwell field and the scalar field decouple from gravity and the problem simplifies in this limit. The black hole solution then becomes a purely gravity one, which is the planar Schwarzschild AdS (SAdS) black hole in 4D i.e. $p = 2$. The metric is given by

$$ds^2 = \left(\frac{r_0}{l}\right)^2 \frac{1}{u^2}(hdt^2 + dx^2 + dy^2) + l^2\frac{du^2}{hu^2} \tag{21.5}$$

where $h = 1 - u^3$, r_0 is the Schwarzschild radius and l is the AdS radius. We introduce the chemical potential since the SAdS black hole has only temperature as a dimensional quantity. The system is then parametrized by μ/T. The AdS boundary is at $u = 0$ and the horizon is at $u = 1$. Then, when $\psi = 0$, the high temperature phase (normal phase) solution [15] becomes

$$A_0 = \mu(1 - u), \quad A_\mu = 0 \tag{21.6}$$

From this equation, we can see that at the horizon, $A_0 = 0$. Thus, due to gauge invariance, the slow fall-off is constant and the fast fall-off represents Coulomb's law in the bulk. As the system approaches the critical temperature T_C, the instability occurs due to the coupling of Maxwell field to the massive scalar field. The effective mass becomes

$$m_{eff}^2 = m^2 - (-g^{00})A_0^2 \tag{21.7}$$

Therefore, for large μ/T, ψ becomes tachyonic (imaginary mass) causing the system to become unstable. Choosing $m^2 = -2/l^2$ means that a certain amount of tachyonic mass is allowed by the AdS space which obeys the Breitenlohner-Freedman (BF) Bound

$$m^2 \geq -\left(\frac{p+1}{2l}\right)^2 \tag{21.8}$$

From [Appendix F], we get the scaling dimensions of the massive scalar field $\psi \sim \psi^(0)u^{\triangle_-} + c_\psi \langle \mathcal{O} \rangle_s u^{\triangle_+}$ as $u \to 0$

$$\triangle_\pm = \frac{p+1}{2} \pm \sqrt{\left(\frac{p+1}{2}\right) + m^2 l^2} \tag{21.9}$$

The external field ψ gives rise to the order parameter $\langle \mathcal{O} \rangle_s$. Substituting Eq. (21.8) in Eq. (20.9), we get the scaling dimensions as

$$\triangle_\pm = (2, 1) \tag{21.10}$$

It is clear that $\triangle_+ + \triangle_- = p+1$. When m^2 is positive, the slow fall-off at the AdS boundary diverges as $\phi \sim u^{\triangle_-}$ which means that the energy momentum tensor is divergent and hence cannot be considered as a perturbation. Thus, the bulk field has two independent solutions: a slow fall-off and a fast fall-off solution. The slow fall-off represents the field which in the dual boundary theory is the external source of the operator $\mathcal{O}$. The fast fall-off represents its response function which is the one-point function $\langle \mathcal{O} \rangle_s$.

In Figure 16, the fall-offs in the thick curve can be considered as operators.

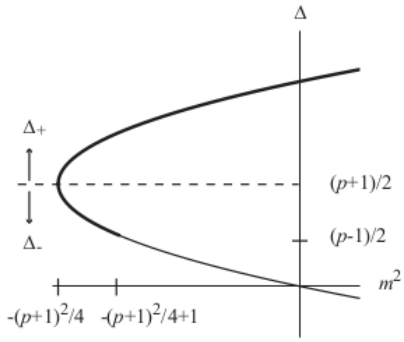

Figure 16: Scaling dimension vs. m^2

21.3 Critical Exponents

In the Ginzburg-Landau (GL) theory, the pseudo free energy density is given by

$$\mathcal{L} = \mathcal{L}_0 + \frac{1}{2}am^2 + \frac{1}{4}bm^4 + ... - mH \tag{21.11}$$

where the order parameter is the magnetization m. When a is positive, the potential becomes minimum at the origin $m = 0$, which gives the behaviour at high temperatures. When a is negative, spontaneous symmetry breaking occurs and the system has a non-zero magnetization, which gives the behaviour at low temperatures. The following are the relations for the critical exponents:

$$\begin{aligned}
\text{Specific heat} &: C_H \propto |T - T_C|^{-\alpha} \\
\text{Spontaneous magnetization} &: m \propto |T - T_C|^{\beta} \\
\text{Magnetic susceptibility} &: \chi \propto |T - T_C|^{-\gamma} \\
\text{Critical isotherm} &: m \propto |H|^{1/\delta} \\
\text{Correlation length} &: \xi \propto |T - T_C|^{-\nu} \\
\text{Static susceptibility} &: \chi(r) \propto r^{-d_s+2-\eta}
\end{aligned}$$

where the last two expressions appear in the inhomogeneous case and d_s is the hyperscaling relation. The first four results satisfy the scaling relations in Eq. (13.6). ν, η are the independent critical exponents. In order to find these exponents, we consider the primary fields

$$\left(\frac{1}{2}, \frac{1}{2}\right) \quad \text{and} \quad (1, 1) \tag{21.12}$$

Following the same procedure as in solving the 2D Ising model, we substitute these fields into the correlation functions with $d_s = 4$ to get the exponents as

$$(\nu, \eta) = \left(\frac{1}{2}, 0\right) \tag{21.13}$$

From the scaling relations, we get all the critical exponents as

$$(\alpha, \beta, \gamma, \delta, \nu, \eta) = \left(0, \frac{1}{2}, 1, 3, \frac{1}{2}, 0\right) \tag{21.14}$$

22 Summary

In Part 1, we have scratched the surface of conformal field theory and have provided an introduction to its application to critical systems by taking the case of two dimensional Ising model and finding its critical exponents at a second order phase transition. Important concepts such as conformal transformations in two dimensions, the Virasoro algebra and the algebra of primary and quasi-primary fields have been studied. Although the conformal symmetry is an approximate symmetry, it has wide applications in various fields of physics.

In Part 2, we have seen the impact of conformal field theory on general relativity giving rise to the powerful tool of AdS/CFT correspondence. Although the gauge/gravity duality relates a bulk quantum theory of gravity to a boundary non-gravitational theory, we have worked in the large-N limit that shows the bulk theory is just classical general relativity. We have derived an explicit dictionary for $\mathcal{N} = 4$ super Yang-Mills theory and IIB string theory. Furthermore, we have explored the application of AdS/CFT in superconductivity. We showed how general relativity can reproduce the basic properties of superconductors. Once we have found the instability that leads to charged scalar hair, we were able to study the phase transition in superconductivity and determine the critical exponents with the help of holographic superconductors.

Furthermore, the AdS/CFT correspondence finds a promising future in condensed matter physics. The anti de-Sitter space/condensed matter physics (AdS/CMT) correspondence aims at applying string theory to condensed matter physics through AdS/CFT. So far, we have been able to describe the transition of a superfluid to an insulator using string theory. Researchers are hopeful that the AdS/CFT correspondence will help describe systems of exotic states of matter in the language of string theory so as to understand those systems in detail. Other areas of physics include nuclear physics, hydrodynamics, etc. where AdS/CFT correspondence is widely applicable.

Appendices

A Operator Algebra

The commutation relations are as follows

$$
\begin{gathered}
[D, P_\mu] = iP_\mu \\
[D, K_\mu] = -iK_\mu \\
[K_\mu, P_\nu] = 2i(g_{\mu\nu}D - L_{\mu\nu}) \\
[K_\rho, L_{\mu\nu}] = i(g_{\rho\mu}K_\nu - g_{\rho\nu}K_\mu) \\
[P_\rho, L_{\mu\nu}] = i(g_{\rho\mu}P_\nu - g_{\rho\nu}P_\mu) \\
[L_{\mu\nu}, L_{\rho\sigma}] = i(g_{\nu\rho}L_{\mu\sigma} + g_{\mu\sigma}L_{\nu\rho} - g_{\mu\rho}L_{\nu\sigma} - g_{\nu\sigma}L_{\mu\rho})
\end{gathered}
$$

As we can see, ignoring D and K_μ, we get back the Poincaré algebra.

B Witt Algebra

The algebra is given by the commutation relations as follows

$$
\begin{aligned}
[l_m, l_n] &= z^{m+1}\partial_z(z^{n+1}\partial_z) - z^{n+1}\partial_z(z^{m+1}\partial_z) \\
&= (n+1)z^{m+n+1}\partial_z - (m+1)z^{m+n+1}\partial_z \\
&= -(m-n)z^{m+n+1}\partial_z \\
&= (m-n)l_{m+n}
\end{aligned}
$$

We arrive at the other two commutators in a similar fashion.

C Central Charge

From Eq. (4.2), we can compute the Jacobi identity

$$\begin{aligned}
[[L_m, L_n], L_0] + [[L_n, L_0], L_m] + [[L_0, L_m], L_n] &= 0 \\
(m-n)cg(m+n, 0) + ncg(n, m) - mcg(m, n) &= 0 \\
(m+n)g(n, m) &= 0
\end{aligned}$$

This implies that $g(n, m)$ vanishes for $m \neq -n$, $|n| \geq 2$. Considering the following Jacobi identity,

$$\begin{aligned}
[[L_{-n+1}, L_n], L_{-1}] + [[L_n, L_{-1}], L_{-n+1}] + [[L_{-}{-1}, L_{-n+1}], L_n] &= 0 \\
(-2n+1)cg(1, -1) + (n+1)cg(n-1, -n+1) + (n-2)cg(-n, n) &= 0
\end{aligned}$$

This leads us to the recursion relation given by

$$\begin{aligned}
g(n, -n) &= \frac{n+1}{n-2} g(n-1, -n+1) \\
&= ... \\
&= \frac{1}{2} {}^{(n+1)}P_3 \\
&= \frac{1}{12(n-2)!} \\
&= \frac{1}{12}(n+1)n(n-1) \\
&= \frac{1}{12}(n^3 - n)
\end{aligned}$$

where $g(n, -n)$ is normalized to -1/2 [3]. Thus, the Virasoro algebra can be given by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

D Energy-Momentum Tensor

For $x^0 = \frac{1}{2}(z+\bar{z})$ and $x^1 = \frac{1}{2i}(z-\bar{z})$, we get the transformation of the energy-momentum tensor as

$$\begin{aligned} T_{zz} &= \frac{\partial x^0}{\partial z}\frac{\partial x^0}{\partial z}T_{01} + \frac{\partial x^0}{\partial z}\frac{\partial x^1}{\partial z}T_{01} + \frac{\partial x^1}{\partial z}\frac{\partial x^0}{\partial z}T_{10} + \frac{\partial x^1}{\partial z}\frac{\partial x^1}{\partial z}T_{11} \\ &= \frac{1}{4}T_{00} - \frac{1}{2}iT_{10} - \frac{1}{4}T_{11} \end{aligned} \tag{6.6}$$

Similarly,

$$T_{\bar{z}\bar{z}} = \frac{1}{4}T_{00} + \frac{1}{2}iT_{10} - \frac{1}{4}T_{11} \tag{6.7}$$

Using Eq. (6.4),

$$T_{z\bar{z}} = T_{\bar{z}z} = \frac{1}{4}(T_{00} + T_{11}) = 0 \implies T_{00} = -T_{11} \tag{6.8}$$

Therefore, we can rewrite

$$T_{zz} = \frac{1}{2}(T_{00} - iT_{10}) \quad \text{and} \quad T_{\bar{z}\bar{z}} = \frac{1}{2}(T_{00} + iT_{10}) \tag{6.9}$$

The two non-vanishing components of the energy–momentum tensor are a chiral field (T_{zz}) and an anti-chiral field ($T_{\bar{z}\bar{z}}$).

E Bekenstein-Hawking Formula

The temperature of Hawking Radiation is given by

$$T = \frac{\hbar\kappa}{2\pi k_B c} = \frac{\hbar c^3}{8\pi G M}$$

where G is Newton's gravitational constant, M is the black hole mass and $\kappa = 1/4GM$ is the surface gravity evaluated at the event horizon of a Schwarzschild black hole. The first law of black hole thermodynamics says that the black hole mass is proportional to the area of the horizon.

$$
\begin{aligned}
dE &= \frac{\kappa c^2}{8\imath G} dA \\
&= \frac{\hbar \kappa}{2\pi k_B c} \frac{k_b c^3}{4G\hbar} dA \\
&= T \frac{k_B c^3}{4G\hbar} dA
\end{aligned}
$$

Comparing the terms with the first law of thermodynamics $dE = TdS$, we conclude that

$$S = \frac{k_B A c^3}{4G\hbar} = \frac{k_B A}{4l_p^2}$$

where $l_p = \sqrt{G\hbar/c^3} \approx 10^{-35} m$ is the Planck length.

F Massive Scalar Field in AdS

The background AdS metric is given by ¡insert eq¿ which has the holographic radial coordinate z. The action for massive scalar field is given by

$$S = -\frac{1}{2} \int d^{d+1}x \sqrt{-g} \left[g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2 \right]$$

Using the on-shell Euler-Lagrangian equation for field theory ¡insert eq¿, we can obtain the equations of motion of the field as

$$\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\mu \phi \right) - m^2 \phi = 0$$

They can be written on the background metric as

$$z^{d+1} \partial_z \left(z^{1-d} \partial_z \phi \right) + z^2 \delta^{\mu\nu} \partial_\mu \partial_\nu \phi - m^2 l^2 \phi = 0$$

On Fourier transforming with respect to the boundary coordinates $x^\mu = (t, \mathbf{x})$,

$$\phi(z, x^\mu) = \int \frac{d^d k}{(2\pi)^d} e^{ik.x} f_k(z)$$

Putting $f_k(z) = z^\triangle$, we get

$$\triangle(\triangle - d) - m^2 l^2 = 0$$

the solution to which are

$$\triangle_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 l^2}$$

These are the scaling dimensions.

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