SEMICLASSICAL GRAVITY THEORY AND QUANTUM FLUCTUATIONS

Project Report

submitted in the fulfillment of the requirement for the degree of

MASTER OF SCIENCE

IN

PHYSICS

by

MANAS BORAH

Reg No:186PH014

DEPARTMENT OF PHYSICS
NATIONAL INSTITUTE OF TECHNOLOGY
KARNATAKA,SURATHKAL,MANGALORE-575025

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DECLARATION

I hereby declare that the report of the P.G. Project Work entitled "SEMICLASSICAL GRAVITY THEORY AND QUANTUM FLUCTUATIONS" which is submitted to National Institute of Technology Karnataka, Surathkal, in the fulfillment of the requirements for the award of the Degree of Master of Science in the Department of Physics, is a bonafide report of the work carried out by me. The material contained in this report has not been submitted to any University or Institution for the award of any degree.In keeping with the general practice in reporting scientific observations, due acknowledgement has been made whenever the work described is based on the findings of other investigators.

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CERTIFICATE

This is to certify that the project entitled "SEMICLASSICAL GRAVITY THEORY AND QUANTUM FLUCTUATIONS" is an authenticated record of work carried out by MANAS BORAH ,Reg.No:186PH014 in the fulfillment of the requirement for the award of the Degree of Master of Science in Physics which is submitted to Department of Physics, National Institute of Technology, Karnataka, during the period 2019-2020.

Dr.DEEPAK VAID
Project Advisor

Chairman-DPGC

ACKNOWLEDGEMENT

I would like to express my profound gratitude to my Project Supervisor Dr. Deepak Vaid, Assistant Professor, Department of Physics, NITK for his constant guidance, encouragement and support. I would like to thank my classmates and the research scholars of the Physics Department for their timely help and valuable support. I acknowledge the library and the internet facilities provided by the institute. I would also like to extend my heartfelt gratitude to my family for all their love and support.

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ABSTRACT

Here, we discuss semiclassical gravity for a scalar field on a flat background in which a classical metric is coupled to the expectation value of the stress tensor. It is argued that this theory is a good approximation only when the fluctuations of the stress tensor are small. The expectation value of the of the stress tensor for a two particle number state , choherent and squeezed states and the criteria for the validity of the semiclassical gravity theory was discussed. [KF93]

Contents

Chapter 1

Introduction

1.1 Why Semiclassical Theory of Gravity ?

We know that classical and quantum mechanics delves into completely opposite sides of the spectrum of dimensions and the fact that classical mechanics can be obtained from quantum mechanics using Ehrenfest's Theorem which will be discussed in section 1.6. We also know that the Einstein's theory of gravity is a classical theory and that, although a lot of research is going on, a complete quantum theory of gravity has not been formulated yet. Since a semiclassical theory seeks to bridge the classical and quantum theories, either of which, can be obtained at particular limits (as it would be seen from sections 1.6 and 1.7), thus the formulation of a semi-classical theory is important.

A natural proposal to describe a gravitational field of a quantum system is the semiclassical gravity theory in which we consider classical spacetime described by the Einstein tensor $G_{\mu\nu}$ and a quantum source (matter or energy), described by the expectation value of the stress-energy tensor $\langle T_{\mu\nu} \rangle$.
In classical gravity theory, the Einstein's theory is given by:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

[Car19]

Thus according to the semiclassical theory the above equation becomes:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \langle T_{\mu\nu} \rangle$$

This would almost certainly fail at Planck scale, where the quantum nature of gravity becomes dominant. However it could also fail away from the Planck scale if the fluctuations of the the stress tensor becomes large. It is seen that the semiclassical theory gives reliable results only when the fluctuations of the stress tensor are not too large, that is, when:

$$\langle T_{\alpha\beta}(x) T_{\mu\nu}(y) \rangle \approx \langle T_{\alpha\beta}(x) \rangle \langle T_{\mu\nu}(y) \rangle$$

On the other hand, if the quantum fluctuations are large, the semiclassical theory cannot be trusted. Here, we will discuss the the issue of the limits of validity of the semiclassical theory, particularly for quantum states in which the expectation value of the local energy density can be negative. [KF93]

1.2 Classical Field Theory

Making a transition from special relativity to General relativity, the metric $\eta_{\mu\nu}$ changes into $g_{\mu\nu}$, which is a dynamical tensor field. General relativity is a special case of classical field theory.

In the case of classical mechanics, a single particle in one dimension is denoted by the position coordinate q(t). The equations of motion for the system is found by using the principle of least action and finding the critical points of the action integral S, given by:

$$S = \int_{t_1}^{t_2} dt L(q(t), \dot{q}(t)) \tag{1.1}$$

Here, the functional $L(q(t), \dot{q}(t))$ is called the Lagrangian of the system. For a point particle system, the Lagrangian has the following form:

$$L(q(t), \dot{q}(t)) = K - V \tag{1.2}$$

Where, K is the Kinetic energy of the point particle and V is the potential energy of the particle. Using the calculus of variations, we find the Euler-Lagrange equations, that are the equations of motion of the single particle, which take the following form:

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \tag{1.3}$$

Now we transition to Field theory. The concept is basically the same but all the coordinates are replaced by a set of spacetime dependent fields $\varphi^i(x^\mu)$, and the action integral becomes a functional of the fields. In field theory, the Lagrangian can be expressed as an integral over a space of a Lagrangian density, L, which is a functional of the fields φ^i and their spacetime derivatives, $\partial_\mu \varphi^i$. The Lagrangian takes the form:

$$L = \int d^3x \mathbf{L}(\varphi^i, \partial_\mu \varphi^i) \tag{1.4}$$

The action integral thus becomes:

$$S = \int dt L(\varphi^i, \partial_\mu \varphi^i) = \int d^4x \mathbf{L}(\varphi^i, \partial_\mu \varphi^i) \tag{1.5}$$

Here, $\mathbf{L}(\varphi^i, \partial_\mu \varphi^i)$ is called the Lagrangian density.
The Lagrangian density is a Lorentz scalar. More convenient to define a field theory by specifying Lagrange density which are functions of φ^i and $\partial_\mu \varphi^i$, from which all of the equations of motion can be derived.
[Car19]

1.3 Dimensional analysis of the Lagrange density

We consider "natural units" in which $c = 1$, $\hbar = k = 1$, where $\hbar = \frac{h}{2\pi}$. Here, h h is the Planck's constant, k is the Boltzmann constant, and c is the speed of light in vacuum. As a result, in natural units, we have:

$$[energy] = [mass] = [length]^{-1} = [time]^{-1}$$

In that case, the action integral is found to be dimensionless.That is M^0.
The volume element has units: $[d^4x] = M^{-4}$
Thus the Lagrange density has the dimensions: $[\mathbf{L}] = M^4$
[Car19]

1.4 Equations of motion

The Euler Lagrange equations are obtained by using the principle of least action, under the small variations of fields:

$$\varphi^i \to \varphi^i + \delta\varphi^i \tag{1.6}$$

$$\partial_\mu \varphi^i \to \partial_\mu \varphi^i + \delta(\partial_\mu \varphi^i) \to \partial_\mu \varphi^i + \partial_\mu \delta\varphi^i \tag{1.7}$$

Thus the variation in the action can be expressed as:

$$\delta S = \delta \int dt L(\varphi^i, \partial_\mu \varphi^i) = \int d^4x \delta \mathbf{L}(\varphi^i, \partial_\mu \varphi^i) \tag{1.8}$$

Now the Lagrange density variation can be described as:

$$\delta\mathcal{L}\left(\phi^{\mathrm{i}}, \partial_\mu \phi^{\mathrm{i}}\right) \to \mathcal{L}\left(\phi^{\mathrm{i}} + \delta\phi^{\mathrm{i}}, \partial_\mu \phi^{\mathrm{i}} + \partial_\mu\left(\delta\phi^{\mathrm{i}}\right)\right) = \mathcal{L}\left(\phi^{\mathrm{i}}, \partial_\mu \phi^{\mathrm{i}}\right) + \frac{\partial\mathcal{L}}{\partial\phi^{\mathrm{i}}}\delta\phi^{\mathrm{i}} + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu \phi^{\mathrm{i}}\right)}\partial_\mu\left(\delta\phi^{\mathrm{i}}\right) \tag{1.9}$$

By simplifying the term, $\left(\frac{\partial \mathbf{L}}{\partial_\mu \varphi^i}\right)(\partial_\mu \delta\varphi^i)$ the difference in Lagrange density becomes:

$$\delta\mathcal{L}\left(\phi^i, \partial_\mu \phi^i\right) = \frac{\partial\mathcal{L}}{\partial\phi^i}\delta\phi^i - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu \phi^i\right)}\right)\delta\phi^i + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu \phi^i\right)}\delta\phi^i\right)$$

The change in action, thus becomes,

$$\delta S = \int d^4x\left(\frac{\partial\mathcal{L}}{\partial\phi^i}\delta\phi^i - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu \phi^i\right)}\right)\delta\phi^i\right) + \int d^4x\left(\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu \phi^i\right)}\delta\phi^i\right)\right) \tag{1.10}$$

The second term goes to 0, since, at the boundaries, $\delta\varphi^i$ goes to 0, so we end up with: For a stationary value, we must have, $\delta S = 0$ Hence, we obtain the Euler-Lagrange equation for a field theory in Minkowski's space-time to be as follows:

$$\frac{\partial L}{\partial\varphi^i} - \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu \varphi^i)}\right) = 0 \tag{1.11}$$

[Car19]

1.5 Einstein's equations using Lagrangian formulation

Einstein's equations can be derived from the principle of least action. We work at spaces of n dimensions (since the results will not depend on the dimensionality). The metric will be considered to have a Lorentzian signature.

We consider a field theory in which the dynamical variables are a set of fields φ^i. The classical solutions to such a theory will be those that are the critical points to the action integral S given by:

$$S = \int dt L(\varphi^i, \partial_\mu \varphi^i) = \int d^n x \mathbf{L}(\varphi^i, \nabla_\mu \varphi^i) \tag{1.12}$$

The covariant derivative is due to the fact that a general space was considered. $d^n x \mathbf{L}(\varphi^i, \nabla_\mu \varphi^i)$ is a tensor denisty. Thus, $\mathbf{L}$ can be written as:

$$\mathbf{L} = \sqrt{(-g)} L_T \tag{1.13}$$

L_T is a scalar. The Lagrange density is applicable whenever the action integral is varied with respect to the metric, on the other hand L_T is applicable in the Euler Lagrange equations:

$$\frac{\partial L}{\partial \varphi^i} - \nabla_\mu \left(\frac{\partial L}{\partial(\partial_\mu \varphi^i)} \right) = 0 \tag{1.14}$$

[Car19]

1.5.1 Construction of the action integral for General Relativity

The dynamical variable for the action integral is taken to be the metric $g_{\mu\nu}$. The independent scalar constructed from the metric, no higher than the second order in its derivatives is the Ricci scalar. The action integral thus formed is called the Hilbert action (or Einstein Hilbert action):

$$S_H = \int d^n x \sqrt{(-g)} R \tag{1.15}$$

Now, applying the variational principle, the integral gets divided into three parts:

$$\delta S_H = \delta S_1 + \delta S_2 + \delta S_3 \tag{1.16}$$

where,

$$\delta S_1 = \int d^n x \sqrt{(-g)} g^{\mu\nu} \delta R_{\mu\nu} \tag{1.17}$$

$$\delta S_2 = \int d^n x \sqrt{(-g)} \delta g^{\mu\nu} R_{\mu\nu} \tag{1.18}$$

$$\delta S_3 = \int d^n x \delta \sqrt{(-g)} g^{\mu\nu} R_{\mu\nu} \tag{1.19}$$

by using the following relations: The Ricci scalar R is given by $R = g^{\mu\nu} R_{\mu\nu}$, where $R_{\mu\nu}$ is the Ricci tensor. The Ricci tensor is obtained from the Riemann tensor as $R_{\mu\nu} = R^\rho_{\mu\sigma\nu}$, and

$$\delta g_{\mu\nu} = -g_{\mu\sigma} g_{\nu\rho} \delta g^{\rho\sigma} \tag{1.20}$$

Thus, the overall variation of the Hilbert action is given by:

$$\delta S_H = \int d^n x \sqrt{-g} (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) \delta g^{\mu\nu} \tag{1.21}$$

The functional derivative of the action is given by:

$$\delta S = \int \sum_{\mathrm{j}} \left(\frac{\delta S}{\delta \varphi^{\mathrm{i}}} \delta \varphi^{\mathrm{i}} \right) \mathrm{d}^{\mathrm{n}} \mathrm{x} \tag{1.22}$$

Thus, we have:

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\phi^i} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \tag{1.23}$$

The above equation is the Einstein's equations in vacuum, since there are no matter fields included. To get the non-vacuum portion of the equation, our action becomes:

$$S = \frac{1}{16\pi G}S_H + S_M \tag{1.24}$$

S_M is the action for matter. Thus, considering the dynamical variable to be the metric $g_{\mu\nu}$ our equation gets modified to:

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}\delta g^{\mu\nu} = \frac{1}{16\pi G}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) + \frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}\delta g^{\mu\nu} = 0 \tag{1.25}$$

Now, we consider the action for a scalar field in curved space:

$$\mathrm{S}_\varphi = \int \left(-\frac{1}{2}\mathrm{g}^{\mu v}\left(\nabla_\mu\varphi\right)\left(\nabla_v\varphi\right) - \mathrm{V}(\varphi)\right)\sqrt{-\mathrm{g}}\mathrm{d}^\mathrm{n}\mathrm{x} \tag{1.26}$$

Varying the action with respect to $g_{\mu\nu}$, we get the Energy momentum tensor to be:

$$\mathrm{T}_{\mu v}^{(\varphi)} = -2\frac{1}{\sqrt{-\mathrm{g}}}\frac{\delta \mathrm{S}_\varphi}{\delta \mathrm{g}^{\mu v}} = \nabla_\mu\varphi\nabla_v\varphi - \frac{1}{2}g_{\mu v}\mathrm{g}^{\rho\sigma}\nabla_\rho\varphi\nabla_\sigma\varphi - \mathrm{g}_{\mu v}\mathrm{V}(\varphi) \tag{1.27}$$

[Car19]

1.6 Ehrenfest's Theorem

If quantum mechanics has to be more general than classical mechanics, we must be able to arrive at classical mechanics at some limiting case. To illustrate the idea, we do the following:
A simple way to calculate the expectation value of momentum is to evaluate the time derivative of $\langle x\rangle$, then multiply with the mass m, which gives us:

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = m\frac{d}{dt}\int_{-\infty}^{\infty} x|\psi|^2 dx = m\int_{-\infty}^{\infty} x\frac{\partial|\psi|^2}{\partial t}dx \tag{1.28}$$

which leads to:

$$\langle p\rangle = -\frac{i\hbar}{2}\int_{-\infty}^{\infty}\left(\psi^*\frac{\partial\psi}{\partial x} - \frac{\partial\psi^*}{\partial x}\psi\right)dx = -i\hbar\int_{-\infty}^{\infty}\psi^*\frac{\partial\psi}{\partial x}dx \tag{1.29}$$

The momentum expectation value can be expressed as:

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = -\mathrm{i}\hbar\int_{-\infty}^{\infty}\psi^*\frac{\partial\psi}{\partial x}dx \tag{1.30}$$

Now, we take the time derivative of the momentum expectation value. Which takes the form

$$\frac{d\langle p\rangle}{dt} = -\int_{-\infty}^{\infty}\frac{dV}{dx}|\psi|^2 dx = -\left\langle\frac{dV}{dx}\right\rangle \tag{1.31}$$

Considering that the potential $V(x)$ is slowly varying, then it can be shown that the expectation values of the momentum and the time derivative of the expectation values of the momentum can be reduced to the Newton's second law if the De Broglie's wavelength associated with the particle is negligible, that is $\lambda \to 0$. [Zet03]

1.7 The Principle behind the WKB Approximation

The Wentzel-Kramer-Brillouin (WKB) method is useful for approximate treatment of systems with slowly varying potentials, that is where the potential remains constant over a single De Broglie wavelength. In classical sustems this property is always satisfied since the De Broglie wavelength tends to zero at that limit as mentioned in the previous section. The WKB approximation can thus be viewed as a semiclassical approximation. [Zet03]

Chapter 2

Introducing Quantum Field Theory

2.1 The harmonic oscillator in a formalism parallel to that used in Quantum Field Theory

We have seen previously, that the action for a particle of mass m, moving in a time dependent potential V(x,t) in 1 dimension is given by:

$$S = \int dt L \tag{2.1}$$

where $\mathrm{L} = \frac{1}{2}\mathrm{m}\dot{\mathrm{x}}^2 - \mathrm{V}(\mathrm{x}, \mathrm{t})$ is the Lagrangian of the system. Thus the equations of motion becomes:

$$m\partial_t^2 x + \partial_x V(x,t) = 0 \tag{2.2}$$

Canonical quantization proceeds by the following processes:
i. Defining the momentum conjugate to x, $p = \frac{\partial L}{\partial(\partial_t x)} = m\partial_t x$
ii. Replacing the dynamic variables x and p by their operators $\hat{x}$ and $\hat{p}$ and
iii. Imposing the canonical commutation relations $[\hat{x}, \hat{p}] = i\hbar$

In the Schrödinger picture, the state is time dependent, the operators are time independent. In the Heisenberg picture, the state is time independent and the operators are time dependent. The commutation relations should hold at each time since the equation of motion implies that if it holds at an initial time, it holds for all times, thus it is still only a single commutation relation. In terms of position and velocity, the commutation relation in the Heisenberg's picture takes the form:

$$[x, \dot{x}] = \frac{i\hbar}{m} \tag{2.3}$$

The equation of motion for a harmonic oscillator with a potential $V(x,t) = \frac{1}{2}m\omega^2(t)x^2$ takes the form:

$$\partial_t^2 x + \omega^2(t)x = 0 \tag{2.4}$$

An operator solution $x(t)$ is considered for this equation. The equation is second order, thus the solution is determined by two Hermitian operators $x(0)$ and $\dot{x}(0)$ and the linear nature of the equation means that the solutions are also linear. The solution $x(t)$ can be written in terms of the non-Hermitian operators x and $x^\dagger$ as:

$$x(t) = f(t)a + \bar{f}(t)a^\dagger \tag{2.5}$$

Here, $f(t)$ is a complex function satisfying the classical equation of motion. $\bar{f}$ is the complex conjugate of f and $a^\dagger$ is the Hermitian conjugate of a.
[Jac05; Lam13]
Using the solution for the harmonic oscillator $x(t) = f(t)a + \bar{f}(t)a^\dagger$ on the commutation relation $[x, \dot{x}] = \frac{i\hbar}{m}$, we find:

$$\langle f, f \rangle [a, a^\dagger] = 1 \tag{2.6}$$

Where the bracket $\langle f, g \rangle$ is given by:

$$\langle f, g \rangle = \frac{im}{\hbar}(\bar{f}\partial_t g - (\partial_t \bar{f})g) \tag{2.7}$$

If the functions f and g are solutions to the harmonic oscillator equation $\partial_t^2 x + \omega^2(t)x = 0$, then the bracket $\langle f, g \rangle$ is independent of time t. Considering the solution f to be chosen so that the real number $\langle f, f \rangle$, then by rescaling, we set, $\langle f, f \rangle$ to be:

$$\langle f, f \rangle = 1 \tag{2.8}$$

Then, the commutation relation, $[a, a^\dagger]$ becomes:

$$[a, a^\dagger] = 1 \tag{2.9}$$

Which is the standard relation seen in Quantum mechanics.

Using the function f and its conjugate $\bar{f}$ and the solution of the harmonic oscillator $x(t)$, using the definition of the bracket, $\langle f, g \rangle$, the operators a and $a^\dagger$ can be found. The relations are:

$$a = \langle f, x \rangle \tag{2.10}$$

and

$$a^\dagger = -\langle \bar{f}, x \rangle \tag{2.11}$$

Both f and x satisfy the equation of motion, thus the brackets mentioned above are time independent. [Jac05]

2.2 Hilbert representation of operators

The Hilbert representation of operators can be built by introducing the normalized state $|0\rangle$ and satisfying the equation $a|0\rangle = 0$. For each n, the state a $|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle$ is a normalized eigenstate of the number operator $N = a^\dagger a$ with eigenvalue n. The span of all the states defined by the state $|n\rangle$ spans the entire Hilbert space of $|0\rangle$ and all the subsequent excitations.
A change in $f(t)$ leads to the change in a that keeps the overall solution $x(t)$ unchanged. The energy is conserved for a special case in which the frequency is constant $\omega(t) = \omega$ and the function $f(t)$ is chosen so that the state $|0\rangle$ is the ground state of the Hamiltonian.
For a solution $x(t) = f(t)a + \bar{f}(t)a^\dagger$ with a general f, the Hamiltonian becomes:

$$H = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2 x^2 \tag{2.12}$$

Using $a|0\rangle = 0$, $a^\dagger|0\rangle = |1\rangle$ and $[a, a^\dagger] = 1$, we find:

$$H = \frac{1}{2}\mathrm{m}\left(\left(\dot{\mathrm{f}}^2 + \omega^2\mathrm{f}^2\right)\mathrm{aa} + \left(\dot{\mathrm{f}}^2 + \omega^2\mathrm{f}^2\right)^*\mathrm{a}^\dagger\mathrm{a}^\dagger + \left(|\dot{\mathrm{f}}|^2 + \omega^2|\mathrm{f}^2\right)\left(\mathrm{aa}^\dagger + \mathrm{a}^\dagger\mathrm{a}\right)\right) \tag{2.13}$$

The above leads to:

$$\mathrm{H}|0\rangle = \frac{1}{2}\mathrm{m}\left(\left(\left(\dot{\mathrm{f}}^2 + \omega^2\mathrm{f}^2\right)^*\right)\mathrm{a}^\dagger\mathrm{a}^\dagger|0\rangle + \left(|\dot{\mathrm{f}}|^2 + \omega^2|\mathrm{f}|^2\right)|0\rangle\right) \tag{2.14}$$

This means that for the state $|0\rangle$ to be an eigenstate of H, the first term must vanish. That is :

$$\dot{f}^2 + \omega^2 f^2 = 0 \tag{2.15}$$

This means that:

$$\dot{f} = \pm i\omega f \tag{2.16}$$

Meaning $f = Ae^{\mp i\omega t}$, hence, for such an f, the norm $\langle f, f\rangle$ is:

$$\langle f, f\rangle = \mp\frac{2m\omega}{\hbar}|f|^2 \tag{2.17}$$

Thus the strictly positive nature of the norm, we have $f = Ae^{-i\omega t}$ which leads to the normalized positive frequency solution to the equation of motion, defined by:

$$f = \sqrt{\frac{\hbar}{2m\omega}}e^{-i\omega t} \tag{2.18}$$

Up to an arbitrary constant phase factor. Using the given f, the Hamiltonian becomes:

$$H = \frac{1}{2}\hbar\omega(aa^\dagger + a^\dagger a) = \hbar\omega(N + \frac{1}{2}) \tag{2.19}$$

The spectrum of the number operator is non negative integers, thus, the minimum energy state is the one with $N = 0$ and the "zero point energy" $\frac{1}{2}\hbar\omega$ It is known that the mean value of position is 0 in the ground state, but the mean of its square is given by:

$$\langle 0|x^2|0\rangle = \frac{\hbar}{2m\omega} \tag{2.20}$$

This characterizes the "zero point fluctuations" of the position in the ground state. [Jac05]

2.3 Quantum scalar field in curved spacetime

The spacetime is taken to be of an arbitrary dimension D, with a metric $g_{\mu\nu}$ of signature $(+, -, -, \ldots, -)$. The action integral for the scalar field ϕ is given by:

$$S = \int d^Dx\sqrt{|g|}\frac{1}{2}\left(g^{\mu v}\partial_\mu\varphi\partial_v\varphi - \left(m^2 + \xi R\right)\varphi^2\right) \tag{2.21}$$

Here, R is the Ricci scalar. [Jac05]

2.3.1 Canonical Quantization

To canonically quantize the field, the Hamiltonian description should be taken. We separate out the space and the time coordinates of the action integral so that it takes the form: $S = \int dx^0 L$ and $\mathrm{L} = \int \mathrm{d}^{\mathrm{D}-1}\mathrm{x}\mathcal{L}$

In the action integral, the Lagrangian, under minimal coupling (R=0) is given by:

$$\mathcal{L} = \sqrt{|\mathrm{g}|}\frac{1}{2}\left(\mathrm{g}^{\mu v}\partial_\mu\varphi\partial_v\varphi - \mathrm{m}^2\varphi^2\right) \tag{2.22}$$

The canonical momentum at a time x^0 is given by:

$$\pi(\mathrm{x}) = \frac{\delta\mathrm{L}}{\delta\left(\partial_0\varphi(\mathrm{x})\right)} = |\mathrm{g}|^{\frac{1}{2}}\mathrm{g}^{\mu 0}\partial_\mu\varphi(\mathrm{x}) = |\mathrm{h}|^{\frac{1}{2}}\mathrm{n}^\mu\partial_\mu\varphi(\mathrm{x}) \tag{2.23}$$

The x labels a point on a surface of constant x^0, the x^0 components of φ is suppressed, n^μ is the unit normal vector to the surface, h is the determinant of the induced spatial metric h_{ij}.

To quantize, the field φ, and its conjugate momentum are considered to be Hermitian operators and satisfy the canonical commutation relation:

$$[\varphi(\mathrm{x}), \pi(\mathrm{y})] = \mathrm{i}\hbar\delta^{\mathrm{D}-1}(\mathrm{x},\mathrm{y}) \tag{2.24}$$

The Dirac delta function on the right hand side is defined by the property:

$$\int \mathrm{d}^{\mathrm{D}-1}y\delta^{\mathrm{D}-1}(\mathrm{x},\mathrm{y})\mathrm{f}(\mathrm{y}) = \mathrm{f}(\mathrm{x}) \tag{2.25}$$

For any scalar function f without the use of any metric volume element. Just like in the previous section, a conserved bracket can be formed from the two complex solutions f and g to the scalar wave equation:

$$\left(|g|^{\frac{-1}{2}}\partial_\mu|g|^{\frac{1}{2}}g^{\mu v}\partial_v + m^2\right)\varphi = 0 \tag{2.26}$$

We the obtain the conserved bracket to be:

$$\langle f, g\rangle = \left(\int d\Sigma_\mu j^\mu\right)_\Sigma \tag{2.27}$$

where,

$$\mathrm{j}^\mu(\mathrm{f},\mathrm{g}) = \frac{\mathrm{i}}{\hbar}|\mathrm{g}|^{\frac{1}{2}}\mathrm{g}^{\mu\nu}\left(\bar{\mathrm{f}}\partial_{\mathrm{v}}\mathrm{g} - \left(\partial_{\mathrm{v}}\bar{\mathrm{f}}\right)\mathrm{g}\right) \tag{2.28}$$

The above mentioned bracket is sometimes called "Klein Gordon inner product" and $\langle f, g\rangle$ called the "Klein Gordon norm" of f. The current density $j^\mu(f, g)$ is divergenceless that is $\partial_\mu j^\mu = 0$ when the functions f and g satisfy the Klein Gordon equation $\left(|g|^{\frac{-1}{2}}\partial_\mu|g|^{\frac{1}{2}}g^{\mu v}\partial_v + m^2\right)\varphi = 0$ hence the value of the integral $\langle f, g\rangle = \left(\int d\Sigma_\mu j^\mu\right)_\Sigma$ is independent of the integrating spacelike surface Σ provided the functions f and g vanish at infinity.

The Klein Gordon inner product satisfies the following:

$\overline{\langle f, g\rangle} = -\langle \bar{f}, \bar{g}\rangle = \langle g, f\rangle$ and $\langle f, \bar{f}\rangle = 0$

It is important to note that the brackets are not positive definite. [Jac05]

2.3.2 Construction of the Hilbert Space

Now, we expand the field operator in terms of the modes and associate the creation and the annihilation operators with the modes. Just as it was done in the harmonic oscillator case, the annihilation operator associated with a complex classical solution f with the field φ can be defined as:

$$a(f) = \langle f, \varphi \rangle \tag{2.29}$$

Similarly, the creation operator can be defined as:

$$a^\dagger(f) = -a(\bar{f}) = -\langle \bar{f}, \varphi \rangle \tag{2.30}$$

Using the definitions, $a(f) = \langle f, \varphi \rangle$, $a^\dagger(f) = -a(\bar{f}) = -\langle \bar{f}, \varphi \rangle$, $\langle f, g \rangle = \left(\int d\Sigma_\mu j^\mu \right)_\Sigma$, where $\mathrm{j}^\mu(\mathrm{f}, \mathrm{g}) = \frac{\mathrm{i}}{\hbar} |\mathrm{g}|^{\frac{1}{2}} \mathrm{g}^{\mu\nu} \left(\bar{\mathrm{f}} \partial_\mathrm{v} \mathrm{g} - \left(\partial_\mathrm{v} \bar{\mathrm{f}} \right) \mathrm{g} \right)$, and from basic field theory, $[\varphi(x), \partial_i \varphi(y)] = 0$ we get the following commutation relations:

i. $[a(f), a^\dagger(g)] = \langle f, g \rangle$

ii. $[a(f), a(g)] = -\langle f, \bar{g} \rangle$

iii. $[a^\dagger(f), a^\dagger(g)] = -\langle \bar{f}, g \rangle$

If f is a positive norm solution with a unit norm $\langle f, f \rangle = 1$ then $a(f)$ and $a^\dagger(f)$ satisfies the usual commutation relation for the raising and lowering operators for a harmonic oscillator. That is:

$$[a(f), a^\dagger(f)] = 1 \tag{2.31}$$

If we have a normalized quantum state $|\Psi\rangle$ which satisfies $a(f)|\Psi\rangle = 0$. This condition satisfies an aspect of the state. That is, for each n, the state $|n, \Psi\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |\Psi\rangle$ is a normalized eigenstate of the number operator $N = a^\dagger(f) a(f)$, with the eigenvalue n.

The span of all these states defines a Fock space of f-wavepacket, "n-particle excitations" above the state $|\Psi\rangle$. [Jac05]

Chapter 3

Semiclassical Gravity Theory and Quantum Fluctuations: Part 1

As stated in the introduction in Chapter 1,
The Einstein's equation is given by:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{3.1}$$

Its semiclassical form, the Einstein's equation becomes:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G \langle T_{\mu\nu}\rangle \tag{3.2}$$

This theory fails in the Planck's scale. It was found that the semiclassical theory give reliable results when the fluctuations in the stress tensor are not too large. That is when:

$$\langle T_{\alpha\beta}(x)T_{\mu\nu}(y)\rangle \approx \langle T_{\alpha\beta}(x)\rangle\langle T_{\mu\nu}(y)\rangle \tag{3.3}$$

The objective of this following calculation is finding the scale at which the semiclassical theory of gravity is no longer reliable. [KF93]

3.1 Quantum states and negative energy density

All forms of matter has non-negative energy density, it is however not true for quantum field theory. Let us consider a massless, minimally coupled scalar field for which the Lagrangian density for Minkowski's spacetime is:

$$\mathcal{L} = \frac{1}{2}\eta^{\mu v}\left(\partial_\mu\varphi\right)\left(\partial_v\varphi\right) \tag{3.4}$$

Thus the stress tensor is given by:

$$\mathrm{T}_{\mu v} = -2\frac{1}{\sqrt{-\mathrm{g}}}\frac{\delta \mathrm{S}_\varphi}{\delta \mathrm{g}^{\mu v}} = \nabla_\mu\varphi\nabla_v\varphi - \frac{1}{2}\mathrm{g}_{\mu v}\mathrm{g}^{\rho\sigma}\nabla_\rho\varphi\nabla_\sigma\varphi \tag{3.5}$$

Thus in Minkowski's spacetime, where we have $g_{\mu\nu} = \eta_{\mu\nu}$ and the conditions mentioned above, we have:

$$\mathrm{T}_{\mu v} = \partial_\mu\varphi\partial_v\varphi - \frac{1}{2}\eta_{\mu v}\eta^{\rho\sigma}\partial_\rho\varphi\partial_\sigma\varphi = \partial_\mu\varphi\partial_v\varphi - \frac{1}{2}\eta_{\mu v}\partial_\sigma\varphi\partial^\sigma\varphi \tag{3.6}$$

Variation of the Lagrangian density with respect to the field φ gives the dynamical equation of motion:

$$(-\partial_t^2 + \nabla^2)\varphi(x) \equiv \partial_\mu \partial^\mu \varphi(x) = 0 \tag{3.7}$$

Thus the quantum field can be expressed as the linear combinations of mode functions as follows:

$$\varphi = \sum_{\mathrm{k}} \left(a_{\mathrm{k}} \mathrm{f}_{\mathrm{k}} + \mathrm{a}_{\mathrm{k}}^{+} \bar{\mathrm{f}}_{\mathrm{k}} \right) \tag{3.8}$$

where, $[a_k, a_{k'}^\dagger] = \delta_{kk'}$, $[a_k, a_{k'}] = 0$ and $[a_k^\dagger, a_{k'}^\dagger] = 0$ The mode function of such a dynamical equation obtained by considering periodic boundary condition of a 3 dimensional box of side L, is given by:

$$f_k = \frac{1}{\sqrt{2L^3\omega}} e^{ik_\alpha x^\alpha} = \frac{1}{\sqrt{2L^3\omega}} e^{\vec{k}.\vec{x} - i\omega t} \tag{3.9}$$

The mode functions are normalized on a spacelike hypersurface under the scalar product:

$$(\varphi_1, \varphi_2) \equiv -\mathrm{i} \int \mathrm{d}^{\mathrm{n}-1}\mathrm{x} \left(\varphi_1(\mathrm{x}) \partial_0 \varphi_2^*(\mathrm{x}) - (\partial_0 \varphi_1(\mathrm{x}) \varphi_2^*(\mathrm{x})) \right) \tag{3.10}$$

such that $[f_k, f_{k'}] = \delta_{kk'}$

Considering the states where a single mode is excited, quantum states of which tale the form:

$$|\Psi\rangle = \sum_{n=1}^{\infty} c_n |n\rangle \tag{3.11}$$

here, $|n\rangle$ is the number eigenstate with n particles in the mode k and c_n are coefficients such that $\sum_{x=1}^{\infty} |c_n|^2 = 1$ Now, we have the stress tensor to be:

$$\mathrm{T}_{\mu v} = \partial_\mu \varphi \partial_v \varphi - \frac{1}{2} \eta_{\mu v} \partial_\sigma \varphi \partial^\sigma \varphi \tag{3.12}$$

And at the mode k, the field in terms of the modes is:

$$\varphi = a_k f_k + a_k^\dagger \bar{f}_k \tag{3.13}$$

And considering the mode function f_k given by:

$$f_k = \frac{1}{\sqrt{2L^3\omega}} e^{ik_\alpha x^\alpha} = \frac{1}{\sqrt{2L^3\omega}} e^{\vec{k}.\vec{x} - i\omega t} \tag{3.14}$$

[KF93]

We find the following:

$$f_k f_k = \frac{1}{2L^3\omega} e^{2i\theta} \tag{3.15}$$

$$f_k^* f_k^* = \frac{1}{2L^3\omega} e^{-2i\theta} \tag{3.16}$$

$$f_k^* f_k = f_k f_k^* = \frac{1}{2L^3\omega} \tag{3.17}$$

The value of the stress tensor becomes:

$$\mathrm{T}_{\alpha\beta} = \frac{1}{2\mathrm{L}^3\omega} \left(\mathrm{k}_\alpha \mathrm{k}_\beta - \frac{1}{2} \eta_{\alpha\beta} \mathrm{k}_\rho \mathrm{k}^\rho \right) \left(-\mathrm{a}_\mathrm{k} \mathrm{a}_\mathrm{k} \mathrm{e}^{2\mathrm{i}\theta} - \mathrm{a}_\mathrm{k}^\dagger \mathrm{a}_\mathrm{k}^\dagger \mathrm{e}^{2\mathrm{i}\theta} + \mathrm{a}_\mathrm{k}^\dagger \mathrm{a}_\mathrm{k} + \mathrm{a}_\mathrm{k} \mathrm{a}_\mathrm{k}^\dagger \right) \tag{3.18}$$

Here, we have $\theta = k_\rho x^\rho$
We consider: $\kappa_{\alpha\beta} = \frac{1}{2\mathrm{L}^3\omega}\left(\mathrm{k}_\alpha \mathrm{k}_\beta - \frac{1}{2}\eta_{\alpha\beta}\mathrm{k}_\rho \mathrm{k}^\rho\right)$
for a massless case, we have, $\kappa_{\alpha\beta} = \frac{1}{2\mathrm{L}^3\omega}$ since $k_\rho k^\rho = 0$ The stress tensor thus becomes:

$$\mathrm{T}_{\alpha\beta} = \kappa_{\alpha\beta}\left(-\mathrm{a_k a_k e^{2i\theta}} - \mathrm{a_k^\dagger a_k^\dagger e^{2i\theta}} + \mathrm{a_k^\dagger a_k} + \mathrm{a_k a_k^\dagger}\right) \tag{3.19}$$

The normal ordered stress tensor in Minkowski's spacetime is the renormalized result obtained by subtracting the Minkowski's vacuum expectation value from the total expectation value. That is:

$$\langle : T_{\alpha\beta} : \rangle = \langle T_{\alpha\beta} \rangle = \langle 0 | T_{\alpha\beta} | 0 \rangle \tag{3.20}$$

Completing the calculation, we get:

$$\langle : T_{\alpha\beta} : \rangle = \kappa_{\alpha\beta}(2n|c_n|^2 - c_{n-2}^* c_n \sqrt{n(n-1)} e^{2i\theta} - c_n^* c_{n-2} e^{-2i\theta}) \tag{3.21}$$

The above calculation can also be done by using the bilinear form:

$$\mathrm{T}_{\alpha\beta}[\mathrm{g}, \mathrm{h}] = \partial_\mu \mathrm{g} \partial_v \mathrm{h} - \frac{1}{2}\eta_{\mu v}\partial_\sigma \mathrm{g} \partial^\sigma \mathrm{h} \tag{3.22}$$

Thus, we get:

$$\mathrm{T}_{\alpha\beta}\left[\mathrm{f_k}, \mathrm{f_k^*}\right] = \mathrm{T}_{\alpha\beta}\left[\mathrm{f_k^*}, \mathrm{f_k}\right] = \kappa_{\alpha\beta} \tag{3.23}$$

$$\mathrm{T}_{\alpha\beta}\left[\mathrm{f_k}, \mathrm{f_k}\right] = -\kappa_{\alpha\beta}\mathrm{e}^{2\mathrm{i}\theta} \tag{3.24}$$

and

$$\mathrm{T}_{\alpha\beta}\left[\mathrm{f_k^*}, \mathrm{f_k^*}\right] = -\kappa_{\alpha\beta}\mathrm{e}^{-2\mathrm{i}\theta} \tag{3.25}$$

Now considering a state composed of two particle number eigenstates,

$$|\Psi\rangle = \frac{1}{\sqrt{1+\epsilon^2}}(|0\rangle + \epsilon|2\rangle) \tag{3.26}$$

where, $|0\rangle$ is the vacuum state satisfying $a|0\rangle = 0$ and $|2\rangle = \frac{1}{\sqrt{2}}a^\dagger$ is the two particle state. ϵ, the relative amplitude of the two states is considered to be real in this case. For this state, similar to the previous calculation, we get the normal ordered expectation value of the stress tensor to be:

$$\begin{aligned}\langle : \mathrm{T}_{\alpha\beta} : \rangle &= \langle \Psi | : \mathrm{T}_{\alpha\beta} : | \Psi \rangle \\ &= \frac{\epsilon}{1+\epsilon^2}\left(\sqrt{2}\left(\mathrm{T}_{\alpha\beta}\left[\mathrm{f_k}, \mathrm{f_k}\right] + \mathrm{T}_{\alpha\beta}\left[\mathrm{f_k^*}, \mathrm{f_k^*}\right]\right) + 2\epsilon\left(\mathrm{T}_{\alpha\beta}\left[\mathrm{f_k}, \mathrm{f_k^*}\right] + \mathrm{T}_{\alpha\beta}\left[\mathrm{f_k^*}, \mathrm{f_k}\right]\right)\right) \\ &= \frac{\kappa_{\alpha\beta}\epsilon}{1+\epsilon^2}(2\epsilon - \sqrt{2}\cos 2\theta)\end{aligned} \tag{3.27}$$

From the above description, it can be seen that the energy density can be positive or negative depending on the value of ϵ and the negative contribution comes from the cross-term and the spacetime dependent phase $\theta = k_\rho x^\rho$. [KF93]

3.2 Discussion regarding Limits of Semiclassical Gravity

The extent to which the semiclassical approximation is violated can be measured by a dimensionless quantity given by:

$$\Delta_{\alpha\beta\mu\nu}(x,y) \equiv \left| \frac{\langle : T_{\alpha\beta}(x) T_{\mu\nu}(y) : \rangle - \langle : T_{\alpha\beta}(x) : \rangle \langle : T_{\mu\nu}(y) : \rangle}{\langle : T_{\alpha\beta}(x) T_{\mu\nu}(y) : \rangle} \right| \tag{3.28}$$

This quantity is a dimensionless measure of the stress tensor fluctuations. (Note that it is not a tensor, but rather the ratio of tensor components.) If its components are always small compared to unity, then these fluctuations are small and we expect the semiclassical theory to hold. [KF93]

Chapter 4

Introduction 2

Previous semester, from the Einstein's equations, the semiclassical limit was taken, which included taking a classical spacetime described by the Einstein tensor $G_{\mu\nu}$ and quantum matter/energy described by the stress energy tensor described by $T_{\mu\nu}$.
The minimally coupled scalar quantum field operator was found from the Klein-Gordon equation in terms of the creation and annihilation operators $a^{\dagger}$ and a respectively. Using the Lagrangian formulation the stress energy tensor was found, which is the converted to the operator form using creation and annihilation operators.
A two particle number state was taken for which, the normalized stress energy tensor was found. For simplicity, the calculation was done in Minkowski's spacetime.

This semester, coherent and squeezed states and their respective operators and parameters will be defined and explored. The normalized stress energy tensor and the fluctuations for coherent and squeezed states will be discussed.

Chapter 5

Coherent States

5.1 Coherent States

Coherent states $|\alpha\rangle$ or also called Glauber states is defined as the eigenstate of the annihilation operator a with eigenvalues α:

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{5.1}$$

since a is a non-hermitian operator, thus the eigenvalue $\alpha = |\alpha|e^{i\varphi}\epsilon\boldsymbol{C}$

5.1.1 Displacement Operator

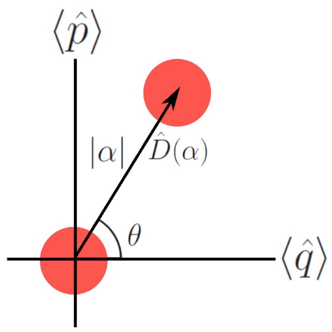

Figure 5.1: Coherent States

The displacement operator $D(\alpha)$ is defined by:

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} \tag{5.2}$$

Here, α is a complex number$\alpha = |\alpha|e^{i\varphi}\epsilon\boldsymbol{C}$, a is the annihilation operator, $a^\dagger$ is the creation operator.

Important Identities of Exponentiated Operators

Considering:

$$e^A = \sum_n \frac{1}{n!}A^n \tag{5.3}$$

We get the following:

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + ... \tag{5.4}$$

$$e^A e^B = e^B e^A e^{-[A,B]} \tag{5.5}$$

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \tag{5.6}$$

From (5.6) we can show that the displacement operator can also be expressed as follows:

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} = e^{-\frac{1}{2}|\alpha|^2 e^{\alpha a^\dagger} e^{-\alpha^* a}} \tag{5.7}$$

Properties of the Displacement Operator

1. $D^{\dagger}(\alpha) = D(-\alpha) = D^{-1}(\alpha)$. It is a unitary operator.
2. $D^{\dagger}(\alpha)aD(\alpha) = a + \alpha$
3. $D^{\dagger}(\alpha)a^{\dagger}D(\alpha) = a^{\dagger} + \alpha^* = (D^{\dagger}(\alpha)aD(\alpha))^{\dagger}$
4. $D(\alpha + \beta) = D(\alpha)D(\beta)e^{-Im(\alpha\beta^*)}$
5. The coherent state $|\alpha\rangle$ is generated from vacuum $|0\rangle$ by the displacement operator $D(\alpha)$

$$D(\alpha)|0\rangle = |\alpha\rangle \tag{5.8}$$

The coherent state as obtained from the action of the displacement operator on vacuum in the phase space is shown in the figure 5.1.
The above properties of the displacement operator can be obtained by using the operator expressions (5.4), (5.5) and (5.6).

5.1.2 Representation in the Number State

The phase of the coherent state $|\alpha\rangle$ describes the wave aspect of the state. To describe the particle aspect, we expand the coherent states in the number state representation as follows:

$$|\alpha\rangle = \sum_n |n\rangle\langle n|\alpha\rangle \tag{5.9}$$

Considering the relations:

$$a^{\dagger}|n\rangle = \sqrt{n+1}|n+1\rangle \tag{5.10}$$

and

$$a|n\rangle = \sqrt{n}|n-1\rangle \tag{5.11}$$

including the properties of the displacement operator which were discussed previously, we can show the number state expansion of the coherent state to be:

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{(\alpha a^{\dagger})^n}{n!}|0\rangle \tag{5.12}$$

5.1.3 Probability distribution of Coherent states

We analyse the probability of detecting a number state $|n\rangle$ in a coherent state $|\alpha\rangle$ as follows:

$$P(n) = |\langle n|\alpha\rangle|^2 = \frac{|\alpha|^{2n}e^{-|\alpha|^2}}{n!} \tag{5.13}$$

The particle number operator is given by:

$$N = a^{\dagger}a$$

The mean particle number is thus given by the expectation value of the particle number operator:

$$\langle n\rangle = \langle\alpha|N|\alpha\rangle = \langle\alpha|a^{\dagger}a|\alpha\rangle = |\alpha|^2 \tag{5.14}$$

this is obtained using (5.1). Thus, (5.13) can be rewritten as follows:

$$P(n) = \frac{\langle n\rangle^n e^{-\langle n\rangle}}{n!} \tag{5.15}$$

Which is a Poissonian distribution. This represents the connection between the mean particle number (particle view) and the complex amplitude squared (wave view).

5.1.4 Orthogonality of Coherent States

The scalar product of coherent states are given by:

$$\langle\beta|\alpha\rangle = \left\langle 0\left|D^\dagger(\beta)D(\alpha)\right|0\right\rangle = \left\langle 0\left|e^{-\beta a^\dagger}e^{\beta^* a}e^{\alpha a^\dagger}e^{-\alpha^* a}\right|0\right\rangle e^{-\frac{1}{2}\left(|\alpha|^2+|\beta|^2\right)} \tag{5.16}$$

The Taylor expansions of the exponentials $e^{-\beta a^\dagger} = \left(1-\beta a^\dagger+\cdots\right)$ and $e^{-\alpha^* a} = (1-\alpha^* a+\cdots)$ annihilate the vacuum state $|0\rangle$. Thus expanding the remaining operators, we get:

$$\begin{aligned}&\left\langle 0\left|\left(1+\beta^* a+\frac{1}{2!}\left(\beta^* a\right)^2+\cdots\right)\left(1+\alpha a^\dagger+\frac{1}{2!}\left(\alpha a^\dagger\right)^2+\cdots\right)\right|0\right\rangle \\ &\qquad = \left(\cdots+\left\langle 2\left|\frac{1}{2!}\sqrt{2!}\left(\beta^*\right)^2+\right\langle 1\right|\beta^*+\langle 0|\right)\left(|0\rangle+\alpha|1\rangle+\frac{1}{2!}\sqrt{2!}\alpha^2|2\rangle+\cdots\right)\end{aligned} \tag{5.17}$$

Using the orthogonality of particle number states given by:

$$\langle n|m\rangle = \delta_{nm}$$

we get:

$$\langle\beta\mid\alpha\rangle = e^{-\frac{1}{2}\left(|\alpha|^2+|\beta|^2\right)}\left(1+\alpha\beta^*+\frac{1}{2!}\left(\alpha\beta^*\right)^2+\cdots\right) = e^{-\frac{1}{2}\left(|\alpha|^2+|\beta|^2\right)+\alpha\beta} \tag{5.18}$$

Thus the transition probability is given by:

$$|\langle\beta|\alpha\rangle|^2 = e^{-|\alpha-\beta|^2} \tag{5.19}$$

This means that coherent states $|\alpha\rangle$ are not orthogonal. They are only so if $|\alpha-\beta| >> 1$

5.1.5 Completeness of Coherent States

Coherent states are not orthogonal and they can be expanded in terms of complete set of states. The completeness condition is given by the following:

$$\frac{1}{\pi}\int d^2\alpha|\alpha\rangle\langle\alpha| = I \tag{5.20}$$

The non orthogonality of coherent states leads to linear dependence of the coherent states, thus the coherent states are "overcomplete".

5.1.6 Uncertainty Principle

The uncertainty principle is about the area in the phase space.
Considering two hermitian non-commuting operators A and B which are canonical conjugates of the other for example $[x,p]$, that satisfies the following relation:

$$[A,B] = iC \tag{5.21}$$

where, C is also a hermitian operator.Then:

$$\Delta A \Delta B \geq \frac{1}{2}|\langle C\rangle| \tag{5.22}$$

Where

$$\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$$

and

$$\Delta B = \sqrt{\langle B^2\rangle - \langle B\rangle^2}$$

The expectation values are computed with respect to the state $|\Psi\rangle$. This state $|\Psi\rangle$ is called a minimum uncertainty state if

$$\Delta A \Delta B = \frac{1}{2}|\langle C\rangle| \tag{5.23}$$

Now, we consider a Harmonic oscillator. Thus, the position and momentum operators are given by:

$$P = i\sqrt{\frac{\hbar\omega}{2}}(a^\dagger - a) \tag{5.24}$$

$$Q = \sqrt{\frac{\hbar}{2\omega}}(a + a^\dagger) \tag{5.25}$$

We find the expectation values with respect to coherent states to be:
$\langle Q\rangle = \langle\alpha|Q|\alpha\rangle$, $\langle Q^2\rangle = \langle\alpha|Q^2|\alpha\rangle$, $\langle P\rangle = \langle\alpha|P|\alpha\rangle$ and $\langle P^2\rangle = \langle\alpha|P^2|\alpha\rangle$ using (4.8).
We the can find the fluctuations of the position ΔQ and momentum ΔP to be as follows:

$$\Delta Q = \sqrt{\langle Q^2\rangle - \langle Q\rangle^2} = \sqrt{\frac{\hbar}{2\omega}} \tag{5.26}$$

and

$$\Delta P = \sqrt{\langle P^2\rangle - \langle P\rangle^2} = \sqrt{\frac{\hbar\omega}{2}} \tag{5.27}$$

From the Heisenberg's uncertainty principle we get:

$$\Delta Q \Delta P = \frac{\hbar}{2} \tag{5.28}$$

which is the lowest possible value for canonically conjugate operators. Thus coherent states are also known as minimum uncertainty states.
How the coherent states become squeezed states, that will be discussed in the following chapter.

Chapter 6

Squeezed States

In the previous chapter, the coherent states and from (5.23) the criterion for a state to be a minimum uncertainty state were discussed. It was seen that coherent states were minimum uncertainty states.
From the previous chapter, we see that for two hermitian non-commuting, canonically conjugate operators A and B, that satisfies the following relation:

$$[A, B] = iC$$

where, C is also a hermitian operator.Then:

$$\Delta A \Delta B \geq \frac{1}{2} |\langle C \rangle|$$

Where

$$\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$$

and

$$\Delta B = \sqrt{\langle B^2 \rangle - \langle B \rangle^2}$$

The expectation values are computed with respect to the state $|\Psi\rangle$. A state is called a squeezed state if one of the observables say A satisfies the following:

$$(\Delta A)^2 < \frac{1}{2} |\langle C \rangle|^2 \tag{6.1}$$

and (5.23) from the previous chapter. Thus for squeezed states, the uncertainty is reduced in one of the variables and amplified for the direction of the operator associated with its conjugate variable. Pure coherent states $|\alpha\rangle$ and number states $|n\rangle$ are not squeezed states.

6.1 Squeezed states

Squeezed state are two parameter coherent states. One is a displacement parameter and the other is the squeezing parameter. They are obtained from coherent states by the application of squeezing operator $S(\varrho)$. It is denoted by $|\alpha, \varrho\rangle$ which is obtained to be:

$$|\alpha, \varrho\rangle = D(\alpha) S(\varrho) |0\rangle \tag{6.2}$$

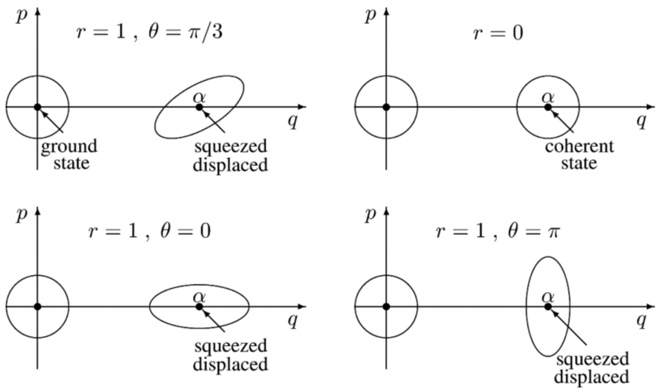

Figure 6.1: How Coherent and Squeezed states behave with phases of their respective parameters

where $D(\alpha)$ and α are the displacement operator and its parameter respectively. $S(\varrho)$ is the squeezing operator and ϱ is its parameter.
The displacement operator is given by:

$$D(\alpha) = e^{\alpha a^{\dagger} - \alpha^* a} = e^{-\frac{1}{2}|\alpha|^2 e^{\alpha a^{\dagger}} e^{-\alpha^* a}} \tag{6.3}$$

where $\alpha = se^{i\gamma}\epsilon \boldsymbol{C}$. The Squeeze operator on the other hand is given by:

$$S(\varrho) = e^{\frac{1}{2}\varrho^* a^2 - \frac{1}{2}\varrho (a^{\dagger})^2} \tag{6.4}$$

where $\varrho = re^{i\delta}\epsilon \boldsymbol{C}$ is the squeezing parameter. From the squeezing operator it is seen that it is involved with the creation or annihilation of an even number of particles. The figure 6.1 represents the different forms of the squeezed state.

6.1.1 Properties of the Squeezing Operator

Using (5.4), (5.5), (5.6), we see that the squeezing operator is a unitary operator and satisfies the following relations:

$$S^{\dagger}(\varrho) a S(\varrho) = a \cosh r - a^{\dagger} e^{i\delta} \sinh r \tag{6.5}$$

and

$$S^{\dagger}(\varrho) a^{\dagger} S(\varrho) = a^{\dagger} \cosh r - a e^{-i\delta} \sinh r = (S^{\dagger}(\varrho) a S(\varrho))^{\dagger} \tag{6.6}$$

A Small Discussion

We consider an operator A and its unitary transformed operator B. They are thus related by:

$$B = P^{\dagger} A P$$

where P is a unitary operator, thus satisfying:

$$P^{\dagger} P = P P^{\dagger} = I$$

We can show that:
$BB = P^{\dagger} AAP$, $BB^{\dagger} = P^{\dagger} AA^{\dagger} P$,$B^{\dagger} B = P^{\dagger} A^{\dagger} AP$ $B^{\dagger} B^{\dagger} = P^{\dagger} A^{\dagger} A^{\dagger} P$
and any other product involving more number of operators.

6.1.2 Some Important Results

From the unitary nature of the squeezing and the displacement operators and their properties (mentioned in the previous chapter), we obtain the following relations:

$$D^\dagger(\alpha)aaD(\alpha) = (a+\alpha)(a+\alpha) \tag{6.7}$$

$$D^\dagger(\alpha)a^\dagger aD(\alpha) = (a^\dagger+\alpha^*)(a+\alpha) \tag{6.8}$$

$$D^\dagger(\alpha)aa^\dagger D(\alpha) = (a+\alpha)(a^\dagger+\alpha^*) \tag{6.9}$$

$$D^\dagger(\alpha)a^\dagger a^\dagger D(\alpha) = (a+\alpha)(a+\alpha) \tag{6.10}$$

$$S^\dagger(\varrho)aaS(\varrho) = (a\cosh r - a^\dagger e^{i\delta}\sinh r)(a\cosh r - a^\dagger e^{i\delta}\sinh r) \tag{6.11}$$

$$S^\dagger(\varrho)a^\dagger a^\dagger S(\varrho) = (a^\dagger\cosh r - ae^{-i\delta}\sinh r)(a^\dagger\cosh r - ae^{-i\delta}\sinh r) = (S^\dagger(\varrho)aaS(\varrho))^\dagger) \tag{6.12}$$

$$S^\dagger(\varrho)a^\dagger a^\dagger S(\varrho) = (a^\dagger\cosh r - ae^{-i\delta}\sinh r)(a^\dagger\cosh r - ae^{-i\delta}\sinh r) = (S^\dagger(\varrho)aaS(\varrho))^\dagger) \tag{6.13}$$

$$S^\dagger(\varrho)aa^\dagger S(\varrho) = (a\cosh r - a^\dagger e^{i\delta}\sinh r)(a^\dagger\cosh r - ae^{-i\delta}\sinh r) \tag{6.14}$$

Similarly, such products of more number of creation operators $a^\dagger$ and annihilation operators can be formed.

6.2 Squeezed Vacuum

Squeezed vacuum is obtained when there is no displacement, only squeezing of the vacuum state $|0\rangle$, which is given by the following:

$$|\varrho\rangle = S(\varrho)|0\rangle \tag{6.15}$$

To show that Squeezed Vacuum is a Squeezed State

We consider the position operator X in terms of the creation and annihilation operator as follows:

$$X = \frac{1}{\sqrt{2}}(a+a^\dagger)$$

For this operator to show that $|\varrho\rangle$ is a squeezed state, we need to show that $(\Delta X)^2 \neq \frac{1}{2}$.

We know that

$$(\Delta X)^2 = \langle X^2\rangle - \langle X\rangle^2$$

We find, from (6.15), that $\langle X\rangle^2 = 0$. Now to calculate $\langle X^2\rangle$ we use (6.5), (6.6), (6.11), (6.12), (6.13) and (6.14) to obtain the following:

$$\langle 0|S^\dagger(\varrho)aaS(\varrho)|0\rangle = -e^{i\delta}\sinh r\cosh r$$

$$\langle 0|S^\dagger(\varrho)a^\dagger a^\dagger S(\varrho)|0\rangle = -e^{-i\delta}\sinh r\cosh r$$

$$\langle 0|S^\dagger(\varrho)a^\dagger aS(\varrho)|0\rangle = \sinh^2 r$$

$$\langle 0|S^\dagger(\varrho)S(\varrho)|0\rangle = 1$$

Thus, from these relations, we get the following:

$$\langle X^2\rangle = \frac{1}{2}(-e^{i\delta}\sinh r\cosh r - -e^{-i\delta}\sinh r\cosh r + 2\sinh^2 r + 1)$$

For $\delta = 0$, we get:

$$\langle X^2\rangle = \frac{1}{2}(-2\sinh r\cosh r + 2\sinh^2 r + 1)$$

For $r > 0$ (for non zero squeezing), we see that always

$$(-2\sinh r\cosh r + 2\sinh^2 r + 1) \neq 1$$

Thus, from the above, we see that:

$$(\Delta X)^2 \neq \frac{1}{2}$$

Which shows that the state $|\varrho\rangle$ is a squeezed state called the squeezed vacuum state.

Considering

$$P = \frac{1}{i\sqrt{2}}(a - a^\dagger)$$

we can show that the state $|\varrho\rangle$ also satisfies the relation:

$$\Delta X\Delta P = \frac{1}{2}$$

Hence, we see that squeezed vacuum states and by extension, general squeezed states, are also minimum uncertainty states, like coherent states.

6.3 Applications of Coherent and Squeezed States

In general, laser light is a coherent state with a very small variance in the photon phase of photons and a large variance in the number of photons in a state. They are related by the uncertainty relation:

$$\Delta N\Delta\Phi \geq \frac{1}{2}$$

Coherent states are applicable in phenomena like quantum hall effect in which, the explanation of edge states is made through coherent states.
Coherent states are also used as the complete basis states in loop quantum gravity.

Squeezed states are used to improve the sensitivity of measurement of devices beyond the quantum noise limits. One such case is with the measurement of laser interferometers.[Sch17]
Two mode squeezed states can be used to make quantitative statements on the strength and nature of correlation between systems.[BP91]

Chapter 7

Semiclassical Gravity Theory and Quantum Fluctuations: Part 2

7.1 Calculation

From the previous chapter, we have seen that a squeezed state can be expressed as:

$$|\alpha, \varrho\rangle = D(\alpha)S(\varrho)|0\rangle$$

where, the displacement and the squeezing operators are given by:

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} = e^{-\frac{1}{2}|\alpha|^2 e^{\alpha a^\dagger} e^{-\alpha^* a}}$$

$$S(\varrho) = e^{\frac{1}{2}\varrho^* a^2 - \frac{1}{2}\varrho (a^\dagger)^2}$$

where, $\alpha = se^{i\gamma} \epsilon \boldsymbol{C}$ and $\varrho = re^{i\delta}\epsilon \boldsymbol{C}$ are the displacement and the squeezing parameters respectively. From (3.19), we have the stress tensor to be:

$$\mathrm{T}_{\alpha\beta} = \kappa_{\alpha\beta}\left(-\mathrm{a_k a_k e^{2i\theta}} - \mathrm{a_k^\dagger a_k^\dagger e^{2i\theta}} + \mathrm{a_k^\dagger a_k} + \mathrm{a_k a_k^\dagger}\right)$$

Thus, with respect to squeezed states, in Minkowski's space, the normal ordered stress tensor becomes:

$$\langle : T_{\alpha\beta} : \rangle = \langle T_{\alpha\beta}\rangle - \langle 0|T_{\alpha\beta}|0\rangle$$

We, thus obtain the expression as follows:

$$\langle \alpha, \varrho \,|: \mathrm{T}_{\alpha\beta} :|\, \alpha, \varrho\rangle = \kappa_{\alpha\beta}\langle 0| : \Big(\mathrm{S}^\dagger(\varrho)\mathrm{D}^\dagger(\alpha) \\ \left(-\mathrm{a_k a_k e^{2i\theta}} - \mathrm{a_k^\dagger a_k^\dagger e^{2i\theta}} + a_k^\dagger a_k + a_k a_k^\dagger\right)\mathrm{D}(\alpha)\mathrm{S}(\varrho)\Big) : |0\rangle \quad (7.1)$$

Using the previous discussion on squeezed states, we get the following result:

$$\langle \alpha, \varrho \,|: \mathrm{T}_{\alpha\beta} :|\, \alpha, \varrho\rangle = 2\mathrm{k}_{\alpha\beta}\left(\sinh^2 \mathrm{r} + \mathrm{s}^2(1-\cos 2(\gamma+\theta)) + \cosh r \sinh r \cos(2\theta+\delta)\right) \quad (7.2)$$

[KF93]

The squeezed vacuum as discussed in section 6.2 corresponds to $\alpha = 0$, $\varrho \neq 0$ and the phase can also be taken to be zero, that is $\delta = 0$. This state in fact is not the vacuum but a superposition of an even number of particles. This is due to the even power of the creation

operator $a^\dagger$ in the squeezing operator.
The energy of the squeezed vacuum state thus becomes:

$$\langle \alpha, \varrho |: \mathrm{T}_{\alpha\beta} :| \alpha, \varrho\rangle = 2\mathrm{k}_{\alpha\beta} \sinh \mathrm{r}(\cosh \mathrm{r} \cos(2\theta) + \sinh \mathrm{r}) \tag{7.3}$$

The vacuum state corresponds to the $r = 0$, and thus the normalize stress tensor vanishes, that is $\langle : T_{\alpha\beta} : \rangle = 0$. The squeezed vacuum, that is for $r \neq 0$, exhibits negative energy states. We see that for a fixed r, as θ goes from 0 to 2π, the energy density becomes negative for a part of the cycle.
Now, we find the expectation value of the squared stress tensor with respect to the squeezed states to be:

$$\begin{aligned} &\langle \alpha, \varrho |: \mathrm{T}_{\alpha\beta}\mathrm{T}_{\mu\nu} :| \alpha, \varrho\rangle = \kappa_{\alpha\beta}\mathrm{K}_{\mu\nu} \\ &\Big(\langle 0| : \Big(S^\dagger(\varrho)D^\dagger(\alpha)\left(-a_k a_k e^{2i\theta} - a_k^\dagger a_k^\dagger e^{2i\theta} + a_k^\dagger a_k + \ \mathrm{a_k a_k^\dagger}\right)\Big(\Big(-\mathrm{a_k a_k e^{2i\theta}} - \mathrm{a_k^\dagger a_k^\dagger e^{2i\theta}} + \mathrm{a_k^\dagger a_k} + \mathrm{a_k a_k^\dagger}\Big) \\ &\qquad D(\alpha)S(\varrho) : |0\rangle \end{aligned} \tag{7.4}$$

Considering the properties of displacement and the squeeze operators and (6.7) to (6.14), the above equation can be simplified to be:

$$\begin{aligned} \langle \alpha, \varrho |: \mathrm{T}_{\alpha\beta}\mathrm{T}_{\mu v} :| \alpha, \varrho\rangle &= 2\kappa_{\alpha\beta}\kappa_{\mu\nu}\left(\mathrm{s}^4[\cos 4(\theta+\gamma) - 4\cos 2(\theta+\gamma)\right. \\ &+ 3] + 3s^2\{2\sinh r\cosh r[2\cos(2\theta+\delta+2\gamma) - \cos(4\theta+\delta+2\gamma)] \\ &+4\sinh^2 r(\cos 2\gamma - \cos 2\theta) - \cos(\delta+2\gamma)\big\} + \\ &\left. 3\sinh^2 r\ \left[\cosh^2 r\cos(4\theta+2\delta) + 3 - 4\cos(2\theta)\right]\right) \end{aligned} \tag{7.5}$$

Again, calculating the same for squeezed vacuum, that is, with $\delta = 0$ and $\alpha = 0$, we get:

$$\begin{aligned} \langle \alpha, \varrho |: \mathrm{T}_{\alpha\beta}\mathrm{T}_{\mu v} :| \alpha, \varrho\rangle &= 2\mathrm{k}_{\alpha\beta}\mathrm{k}_{\mu \mathrm{v}} \sinh^2 \mathrm{r}\left(2\cosh^2 \mathrm{rcos}4\theta\right. \\ &\left.-8\sinh r\cosh r\cos(2\theta) + 3\left(\sinh^2 r + \cosh^2 r\right)\right) \end{aligned} \tag{7.6}$$

[KF93]

7.2 Discussion

The amount of squeezing may be measured by the squeezing parameter $|\varrho| = r$. When there is no squeezing, that is $\varrho = 0$, the state gets converted to a coherent state. In this particular case, we get:

$$\langle T_{\alpha\beta}(x)T_{\mu\nu}(y)\rangle = \langle T_{\alpha\beta}(x)\rangle\langle T_{\mu\nu}(y)\rangle = 2\kappa_{\alpha\beta}\kappa_{\mu\nu}s^4(\cos 4(\theta+\gamma) - 4\cos(2\theta+2\gamma) + 3) \tag{7.7}$$

Thus, From (3.28) we get, for a coherent state,

$$\Delta_{\alpha\beta\mu\nu}(x, y) = \Delta = 0 \tag{7.8}$$

From this criterion, we see that semiclassical gravity theory is a good approximation for coherent states.

From the squared stress energy tensor of the squeezed vacuum state given in (5.6), when $r = 0$, that is the vacuum state, we find $\langle : T_{\alpha\beta}(x)T_{\mu\nu}(y) : \rangle = 0$ as expected.
We, know that for coherent states, $\Delta = 0$. For states sufficiently close to coherent states, $\Delta << 1$. As the magnitude of the squeezing parameter r increases relative to α, Δ also increases. We see that as we squeeze a state so as to approach $\rho = \langle : T_{00} \rangle < 0$, we get $\Delta \to 1$.
Thus squeezed states for which energy density is negative, exhibit large energy fluctuations. Hence Semiclassical gravity is not a good approximation for such states. [KF93]

Chapter 8

Conclusion

In the previous semester:

This project required basics of classical and quantum field theory and some fundamental knowledge of General Theory of Relativity. A considerable amount of time was spent in studying and understanding the conceptual development of General Theory of Relativity up to the emergence of Einstein's equations through classical field theory. How the classical gravity theory transitioning to the semiclassical theory for a scalar field in Minkowski's space was also studied for a number state. This was further explored for a two particle number state.

In the current semester:
Coherent and squeezed states, their operators and their parameters were defined and their properties were explored. The normalized stress energy tensor and the squared stress energy for the coherent and the squeezed states were found. The compatibility of the semiclassical gravity theory and coherent/squeezed states were discussed.

This calculation can be further extended to Casimir vacuum energy. Here, we can find that the dimensionless Casimir energy density fluctuations are at least of the order of unity, which would bring us to the conclusion that Casimir energy cannot be described by a fixed classical metric. [KF93]

Chapter 9

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