ELECTRON ACCELERATION IN CURVED SPACETIME

Project Report

submitted in partial fulfillment of the requirement for the degree of

MASTER OF SCIENCE

IN

PHYSICS

by

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DECLARATION

I hereby declare that the report for the post-graduate project work entitled "ELECTRON ACCELERATION IN CURVED SPACETIME" which is submitted to National Institute of Technology Karnataka Surathkal, in partial fulfillment of the requirements for the award of the Degree of Master of Science in the Department of Physics, is a bonafide report of the work carried out by me. The material contained in this report has not been submitted to any University or Institution for the award of any degree.

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Date: January 2021

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CERTIFICATE

This is to certify that the project entitled "ELECTRON ACCELERATION IN CURVED SPACETIME" is an authenticated record of work carried out by ANAND MENON, Roll No.: 196PH003 in partial fulfillment of the requirement for the award of the Degree of Master of Science in Physics which is submitted to Department of Physics, National Institute of Technology Karnataka, Surathkal, during the period 2020-2021.

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ABSTRACT

This project report starts with giving a description of the Dirac equation, including an account of Lie groups and Lie algebra, and the representation of Lorentz group. This also includes a brief illustration of the Heisenberg picture. Then, a detailed mathematical analysis on the Schwarzschild solution of the Einstein field equation is done. It involves, first determining the characteristics of the spacetime interval of a spherically symmetric and static field, and then using it to solve the Einstein field equations, by first evaluating the Christoffel symbols, the Ricci tensor and the Ricci scalar from the metric obtained from that spacetime interval. The metric obtained from the Schwarzschild solution will be used to define the spacetime in which the electron accelerates.

Following the Schwarzschild solution, a comprehensive account of tetrad formalism is done, while also exploring the transformation properties of the Dirac equation. This section will specifically deal with how the tetrad formalism has been used to make the Dirac equation, suitable for use in curved spacetime. Spin connection values for the spacetime metric, that is used for determining the acceleration of the electron (that has been calculated in the previous section using the Schwarzschild metric), have also been evaluated.

The final section of the report comprises, first of the reducing the Dirac equation that consisted of a four-component spinor, into two equations involving two-component spinors. It also consists of an account of the Lorentz transformation of the spin components, by taking an analogy of it with the Lorentz transformation of the electric and magnetic fields. Using this, the properties of the spin components have been determined. This is then followed by using an iterative approach, in which the two equations obtained earlier are then condensed into a non-relativistic Schrödinger-like equation, from which the Hamiltonian is obtained and then the velocity and the acceleration of the electron is calculated.

Contents

Chapter 1

Objectives

Quantum Field theory started with a set of theoretical rules that was built analogous to quantum mechanics. It was one of the earliest attempts made to combine classical field theory, special relativity and quantum mechanics. It has been used in particle physics to construct physical models of subatomic particles, in condensed matter physics to develop the models of quasi-particles, in the development of Standard models, etc.

In this project, I have made an attempt to understand how the Dirac equation was developed from the representation theory of groups. Therefore, one of the main objectives was to account for the nature of the Dirac equation, and what makes it unique and its significance in the realm of quantum field theory. I have also made an attempt in understanding Lie groups and Lie algebra, as well as the representation of Lorentz group.

Another objective was to find out how the Dirac equation can be used to find the acceleration of an electron in curved spacetime. Thus, an elaborate study on the Schwarzschild solution of the Einstein field equation, which approximately defines the spacetime around slowly-rotating objects like planets and stars, and the Tetrad formalism was to be done. The final objective was to compare the result obtained using the analytical calculation of the Dirac equation, and the result obtained in the case of Newtonian gravitation.

Chapter 2

Introduction to Dirac Equation

2.1 Heisenberg picture

The Schrödinger picture allowed us to implement translations quantum-mechanically, using,

$$U(a) = e^{-ia_\mu \mathbf{P}^\mu}$$

The above operation is Hermitian and unitary.
In the Heisenberg picture, the operator is defined as,

$$\mathcal{O}_H = e^{iHt}\mathcal{O}e^{-iHt} \tag{2.1}$$

Now, differentiating the equation (2.1) with respect to t,

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{O}_{\mathcal{H}}}{\mathrm{d}t} &= \frac{\mathrm{d}\big(e^{iHt}\mathcal{O}e^{-iHt}\big)}{\mathrm{d}t} \\
&= i[H\mathcal{O}_H - \mathcal{O}_H H] \\
\therefore \frac{\mathrm{d}\mathcal{O}_H}{\mathrm{d}t} &= i[H, \mathcal{O}_H]
\end{aligned}$$

In Heisenberg picture, the commutator relations between the field operator $\phi(t, \mathbf{x})$ and its conjugate canonical momentum operator $\pi(t, \mathbf{x})$ are as follows,

$$[\phi(t, \mathbf{x}), \phi(t, \mathbf{y})] = [\pi(t, \mathbf{x}), \pi(t, \mathbf{y})] = 0$$

We have used time as an argument in ϕ and π, since we are in the Heisenberg picture. In Heisenberg picture, it is easier to describe spatially local processes since the field operator $\phi(t, \mathbf{x})$ is defined at a point in spacetime. Now we define the equations of motion for the spatially local operations, $\phi(t, \mathbf{x})$ and $\pi(t, \mathbf{x})$ as,

$$\begin{aligned}
\frac{\mathrm{d}\phi(t, \mathbf{x})}{\mathrm{d}t} &= i[H, \phi(t, \mathbf{x})] \\
&= \frac{i}{2}\left[\int d^3x' \pi^2(t, \mathbf{x}') + \big(\boldsymbol{\nabla}\phi(t, \mathbf{x}')\big)^2 + m^2\phi^2(t, \mathbf{x}'), \phi(t, \mathbf{x})\right]
\end{aligned}$$

where the Hamiltonian,

$$H = \int d^3x' \pi^2(t, \mathbf{x}') + \big(\boldsymbol{\nabla}\phi(t, \mathbf{x}')\big)^2 + m^2\phi^2(t, \mathbf{x}')$$

corresponds to the Hamiltonian of the single free field $\phi(t,\mathbf{x})$. Since $\phi(t,\mathbf{x}')$ and $\phi(t,\mathbf{x})$ commute with each other,

$$\Big[\big(\boldsymbol{\nabla}\phi(t,\mathbf{x}')\big)^2,\phi(t,\mathbf{x})\Big]=\big[\phi^2(t,\mathbf{x}'),\phi(t,\mathbf{x})\big]=0$$

Then, the above equation becomes,

$$\begin{aligned}\therefore\ \frac{\mathrm{d}(\phi(t,\mathbf{x}))}{\mathrm{d}t}&=\frac{i}{2}\int d^3x'\big[\pi^2(t,\mathbf{x}'),\phi(t,\mathbf{x})\big]\\&=-i\int d^3x'\pi(t,\mathbf{x}')i\delta^3(-\mathbf{x}+\mathbf{x}')\\&=\pi(t,\mathbf{x})\end{aligned}\tag{2.2}$$

Further, if we try to find the derivative of $\pi(t,\mathbf{x})$,

$$\begin{aligned}\frac{\mathrm{d}(\pi(t,\mathbf{x}))}{\mathrm{d}t}&=i[H,\pi(t,\mathbf{x})]\\&=i\int d^3x'\Big[\pi^2(t,\mathbf{x}')+\big(\boldsymbol{\nabla}\phi(t,\mathbf{x}')\big)^2+m^2\phi^2(t,\mathbf{x}'),\pi(t,\mathbf{x})\Big]\end{aligned}$$

Since $\pi(t,\mathbf{x})$ and $\pi(t,\mathbf{x}')$ commute with each other, then,

$$\big[\pi^2(t,\mathbf{x}'),\pi(t,\mathbf{x})\big]=0$$

$$\begin{aligned}\therefore\ \frac{\mathrm{d}(\pi(t,\mathbf{x}))}{\mathrm{d}t}&=i\int d^3x'\Big[\big(\boldsymbol{\nabla}\phi(t,\mathbf{x}')\big)^2+m^2\phi^2(t,\mathbf{x}'),\pi(t,\mathbf{x})\Big]\\&=-\int d^3x'\big[\boldsymbol{\nabla}\phi(t,\mathbf{x}')\boldsymbol{\nabla}\big(\delta^3(\mathbf{x}'-\mathbf{x})\big)\big]-m^2\int d^3x'\phi(t,\mathbf{x})\delta^3(\mathbf{x}'-\mathbf{x})\\&=\Big[\boldsymbol{\nabla}\phi(t,\mathbf{x}')\boldsymbol{\nabla}\big(\delta^3(\mathbf{x}'-\mathbf{x})\big)\Big|_{-\infty}^{\infty}+\int d^3x'\big[\nabla^2\phi(t,\mathbf{x}')\big(\delta^3(\mathbf{x}'-\mathbf{x})\big)\big]-m^2\phi(t,\mathbf{x})\\&=\nabla^2\phi(t,\mathbf{x})-m^2\phi(t,\mathbf{x})\end{aligned}\tag{2.3}$$

But from equation (2.2),

$$\frac{\mathrm{d}(\pi(t,\mathbf{x}))}{\mathrm{d}t}=\frac{\mathrm{d}^2(\phi(t,\mathbf{x}))}{\mathrm{d}t^2}\tag{2.4}$$

Equating equation (2.3) and equation (2.4) and rearranging, we get,

$$\begin{aligned}&\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}-\nabla^2\right)\phi(t,\mathbf{x})+m^2\phi(t,\mathbf{x})=0\\\text{or,}\quad&\partial^\mu\partial_\mu\phi(t,\mathbf{x})+m^2\phi(t,\mathbf{x})=0\end{aligned}\tag{2.5}$$

The above equation is called the Klein-Gordon equation.

2.2 Lorentz Invariance in wave equation

If ϕ is a field or a collection of fields and D is some differential operator, then if $\phi(x)$ satisfies the equation $D\phi(x)=0$, and if we perform a rotation or boost to the frames of reference, and then if the new transformed field in the new reference field satisfies the same equation, then $D\phi(x)=0$ is said to be relativistically invariant.
An equation of motion is Lorentz invariant, if it follows from a Lagrangian that is a Lorentz

scalar. This is a result of Principle of least action.
Consider an arbitrary Lorentz transform as,

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

Where Λ is a 4x4 matrix.
Consider $\phi(x)$ as a local value of some quantity distributed through space. Let the field, $\phi(x)$, have a maximum value at $x = x_0$.
After Lorentz transformation, the new transformed field will have a maxima at $x = \Lambda x_0$.
The field will transform as,

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$$

We will now, consider the Lorentz transformation of Klein-Gordon Lagrangian,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2$$

The mass term will transform as $\frac{1}{2}m^2\phi^2(\Lambda^{-1}x)$.
The $\partial_\mu\phi$ will transform as,

$$\partial_\mu\phi(x) \to \partial_\mu\phi(\Lambda^{-1}x)$$

Now,

$$\partial_\mu\phi(\Lambda^{-1}x) \equiv \frac{\partial\phi(\Lambda^{-1}x)}{\partial x'^\mu}$$

But $x'^\mu = \Lambda^\mu{}_\nu x^\nu$. Therefore, we can write the above expression as,

$$\begin{aligned}\partial_\mu\phi(x) \to \partial_\mu\big(\phi(\Lambda^{-1}x)\big) &= \big(\Lambda^{-1}\big)^\mu{}_\nu \frac{\partial\phi(\Lambda^{-1}x)}{\partial x^\nu} \\ &= \big(\Lambda^{-1}\big)^\mu{}_\nu (\partial_\nu\phi)(\Lambda^{-1}x)\end{aligned}$$

The kinetic energy term in the Klein-Gordon Lagrangian transforms as,

$$\begin{aligned}[\partial_\mu\phi(x)]^2 &\to \big[\partial_\mu\phi'(x)\big]^2 \\ &= g^{\mu\nu}\big[\partial_\mu\phi'(x)\big]\big[\partial_\nu\phi'(x)\big] \\ &= g^{\mu\nu}\Big[\big(\Lambda^{-1}\big)^\alpha{}_\mu \partial_\alpha\phi\Big]\Big[\big(\Lambda^{-1}\big)^\beta{}_\nu \partial_\beta\phi\Big]\big(\Lambda^{-1}x\big) \\ &= g^{\alpha\beta}(\partial_\alpha\phi)(\partial_\beta\phi)\big(\Lambda^{-1}x\big) \\ &= (\partial_\mu\phi)^2\big(\Lambda^{-1}x\big)\end{aligned}$$

Thus, the Lagrangian transforms like a scalar,

$$\mathcal{L}(x) \to \mathcal{L}(\Lambda^{-1}x)$$

Therefore, the Klein-Gordon Lagrangian is Lorentz invariant. The Action, S, is also Lorentz invariant. It is defined as,

$$S = \int \mathcal{L} d^4x$$

The equation of motion corresponding to the Klein-Gordon Lagrangian is also Lorentz invariant. The Klein-Gordon equation of motion is given by,

$$\big[\partial^2 + m^2\big]\phi(x) = 0$$

The transformed equation of motion can be written as,

$$\begin{aligned}\left[\partial^2 + m^2\right]\phi'(x) &= \left[g^{\mu\nu}\partial_\mu\partial_\nu + m^2\right]\phi'(x) \\ &= \left[g^{\mu\nu}\left(\Lambda^{-1}\right)^{\rho}{}_{\mu}\partial_\rho\phi\left(\Lambda^{-1}\right)^{\sigma}{}_{\nu}\partial_\sigma\phi + m^2\right]\phi(\Lambda^{-1}x) \\ &= \left[g^{\rho\sigma}\partial_\rho\partial_\sigma + m^2\right]\phi(\Lambda^{-1}x) \\ &= \left[\partial^2 + m^2\right]\phi\left(\Lambda^{-1}x\right)\end{aligned}$$

Here, we have considered the transformation of a single component scalar field, ϕ. In case of a multi-component field, the transformation becomes more complex. The most familiar is the vector field like 4-current density, $j^\mu(x)$ or the vector potential, $A^\mu(x)$. Here, orientation of the quantity in spacetime must also be boosted. Tensors of arbitrary rank can be built out of vectors by adding more indices and corresponding more factors of Λ in the transformation law. In general, any equation in which each term has the same set of uncontracted Lorentz indices will be Lorentz invariant.
Thus, tensor notation gives a large class of Lorentz-invariant equations. To find them, we can consider all possible transformations of a field. For simplicity, we will consider linear transformation. So if Φ_a is an n-component multiplet, the Lorentz transformation will be a n×n matrix,

$$\Phi_a(x) \to M_{ab}\Phi_b(\Lambda^{-1})x$$

This can be simplified as,

$$\Phi \to M(\Lambda)\Phi$$

Now, if we consider two successive Lorentz transformation Λ and Λ', the net result must be another Lorentz transformation, Λ''. Thus, Lorentz transformation forms a group. This condition must also be satisfied by $M(\Lambda)$, that is,

$$\Phi \to M(\Lambda')M(\Lambda)\Phi = M(\Lambda'')\Phi \qquad (\text{For} \quad \Lambda'' = \Lambda'\Lambda)$$

Therefore, the correspondence between M and Λ must be preserved under multiplication. Here, M must form an n-dimensional representation of Lorentz group. In a continuous group, transformations that are infinitesimally close to the identity define the Lie algebra of the group. The Lie algebra is a vector space and the basis vectors spanning this vector space is called the generators of Lie algebra. By finding the matrix representation of the generators of the group, we can find the matrix representation of the continuous group.

2.3 Lie Groups and Lie Algebra

The main idea in Lie algebra, is to study a continuous group that depends on a parameter (considered to be a manifold structure) is to study symmetries in the group, very close to the identity, and use these elements to describe larger elements, by exponential. Here, we suppose the group to simultaneously be a manifold and that the group operations are continuous. There will be a distinguished point in this manifold representing the identity (since all groups should have an identity). The operations occur via multiplication,

$$g \times h \to g \cdot h \forall g, h \in \mathcal{M}$$

These operations are continuous, including the inverse operation,

$$g^{-1} \times g \to \mathbb{I}$$

Let us consider the following manifold,

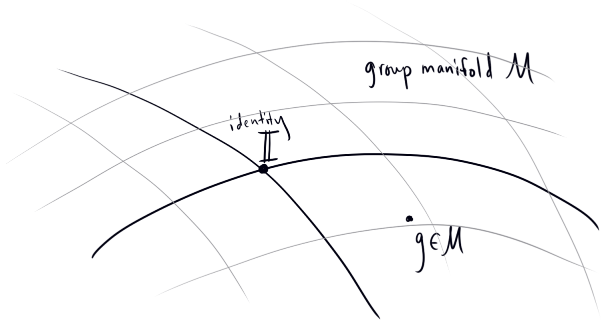

Figure 2.1: Sketch of a group manifold with identity $\mathbb{I}$ and an element of the manifold g.

The theory of Lie group representations is to study elements very close to the identity. So we consider a tangent plane to the identity and study the group elements within the infinitesimal small neighbourhood of $\mathbb{I}$. Taylor's theorem states that the closer we get to $\mathbb{I}$, the better is the tangent plane approximation to the manifold.

$$g - \mathbb{I} \sim O(\epsilon)$$

Where, ϵ is infinitesimal and $g \in \mathcal{M}$, and is infinitesimally close to $\mathbb{I}$.
The space of such infinitesimal elements is considered to be the Tangent space, $T_{\mathbb{I}}\mathcal{M}$. This is a linear space, and is called Lie Algebra. Any element of $T_{\mathbb{I}}\mathcal{M}$ determines an element of $\mathcal{M}$. In other words, an algebra structure on the tangent space gives the Lie group structure and vice-versa. The basis vectors for $T_{\mathbb{I}}\mathcal{M}$ is $X^j, j = 1, 2, 3, ..., dim(\mathcal{M})$.

Let $X \in T_{\mathbb{I}}\mathcal{M}$. Then we build $\mathbb{I} + \epsilon X \simeq e^{\epsilon X} \in \mathcal{M}$ (upto $\mathcal{O}(\epsilon)$). Here, $\mathbb{I} + \epsilon X$ is an element of the first group upto the first order. Since multiplication is continuous, we can define,

$$g(s) \equiv \left(\mathbb{I} + \frac{s}{n}X\right)^n$$

As n becomes bigger, $\frac{s}{n} \to \epsilon$ and it becomes an element in a Lie group.

$$\therefore g(s) \equiv \lim_{n\to\infty} \left(\mathbb{I} + \frac{s}{n}X\right)^n = e^{sX}$$

Thus, every element X of the tangent space of $\mathbb{I}$, determines, by the repeated product of $\left(\mathbb{I} + \frac{s}{n}X\right)^n$, an element of the Lie group.
We will now find an algebra structure in the tangent space. Since $\mathcal{M}$ is a group, we know,

$$[g, h] \equiv ghg^{-1}h^{-1} \in \mathcal{M} \tag{2.6}$$

Thus, $[\cdot, \cdot] : \mathcal{M} \times \mathcal{M}$.
This is called the Lie bracket and this mapping turns the tangent space into a Lie algebra, and $[\mathbb{I}, \mathbb{I}] = \mathbb{I}$. Also, $[\cdot, \cdot] : T_{\mathbb{I}}\mathcal{M} \times T_{\mathbb{I}}\mathcal{M} = T_{\mathbb{I}}\mathcal{M}$.
Now instead of g and h, we substitute $(\mathbb{I} + \epsilon A)$ and $(\mathbb{I} + \delta B)$, respectively.

$$\begin{aligned}[(\mathbb{I} + \epsilon A), (\mathbb{I} + \delta B)] &= (\mathbb{I} + \epsilon A)(\mathbb{I} + \delta B)(\mathbb{I} - \epsilon A)(\mathbb{I} - \delta B) \\ &= \mathbb{I} + \epsilon\delta(AB - BA) + \mathcal{O}(\epsilon^2) + \mathcal{O}\big(\delta^2\big)\end{aligned}$$

We learn that, $[A, B] = AB - BA \in T_{\mathbb{I}}\mathcal{M}$.

2.4 Infinitesimal Lorentz transformations

The infinitesimal Lorentz transformation is written as,

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}$$

Here, $\omega^{\mu}{}_{\nu}$ is the infinitesimal factor, and $\Lambda^{\mu}{}_{\nu}$ is the infinitesimal Lorentz transform which satisfies the relation,

$$\Lambda^T \eta \Lambda = \eta$$

Where, η is the Minkowski metric. Therefore, we can write the above expression using the infinitesimal Lorentz transformation as,

$$(\delta^{\mu}{}_{\sigma} + \omega^{\mu}{}_{\sigma})\eta^{\sigma\tau}(\delta^{\nu}{}_{\tau} + \omega^{\nu}{}_{\tau}) = \eta^{\mu\nu}$$
$$\therefore \omega^{\mu\nu} + \omega^{\nu\mu} + \eta^{\sigma\tau}\omega^{\mu}{}_{\sigma}\omega^{\nu}{}_{\tau} = 0$$

Since $\omega^{\mu}{}_{\nu}$ is already infinitesimal in its own accord, we can ignore the $\eta^{\sigma\tau}\omega^{\mu}{}_{\sigma}\omega^{\nu}{}_{\tau}$, and we can write,

$$\omega^{\mu\nu} + \omega^{\nu\mu} = 0 \tag{2.7}$$

This implies that $\omega^{\mu\nu}$ is anti-symmetric. Thus, $\omega^{\mu\nu}$ will have 6 independent degrees of freedom, or, 6 infinitesimal symmetries, or, 6 conserved currents.
The infinitesimal Lorentz transformation of a field is given by,

$$\phi(x) \to \phi'(x) \equiv \phi(\Lambda^{-1}x) \equiv \phi(x^{\mu} - \omega^{\mu}{}_{\nu}x^{\nu})$$

2.5 Lorentz group

The Lorentz group is defined as,

$$G = \left\{\Lambda \middle| \eta_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma}\right\}$$

Where, Λ is a 4x4 matrix called the Lorentz boost metric.
The tangent space to the identity of this group is,

$$T_{\mathbb{I}}G = \left\{\omega \middle| \eta_{\mu\nu}(\mathbb{I} + \epsilon\omega)^{\mu}_{\rho}(\mathbb{I} + \epsilon\omega)^{\nu}_{\sigma} = \eta_{\rho\sigma}\right\}$$
$$= \left\{\omega \middle| \omega^{\mu\nu} = -\omega^{\nu\mu}\right\}$$

Note that the $\omega^{\mu\nu}$ is an anti-symmetric matrix with covariant indices.
We choose the basis, 'J^A' for the tangent space, such that,

$$\left\{\omega \middle| \omega = \sum_{A=1}^{6} \frac{\Omega_A}{2} J^A\right\}$$

The index A can be changed to a double index $\rho\sigma$ such that $0 < \rho < \sigma \leq 3$. Thus, the elements of this space can then be explicitly written using this basis as,

$$T_{\mathbb{I}}G = \left\{\omega \middle| \omega = \sum_{\rho\sigma} \frac{1}{2}\Omega_{\rho\sigma}J^{\rho\sigma}\right\}$$

Where,

$$[J^{\rho\sigma}]^{\mu}{}_{\nu} = \eta^{\rho\mu}\delta^{\sigma}{}_{\nu} - \eta^{\sigma\mu}\delta^{\rho}{}_{\nu} \tag{2.8}$$

And $J^{\rho\sigma}$ is called the Generator of the Lorentz group.
The Generators of boost are J^{01}, J^{02} and J^{03}. Exponentiating these generators will give us a 4×4 matrices that are pure Lorentz boost. The other three remaining generators are generators of rotation, J^{12}, J^{13} and J^{23}. Exponentials of these generators will give us 4×4 matrices that give rotation. Exponentiating any six of these generators or its linear combinations, we can get any Lorentz transformation.
The Lie bracket for the generator of this group is given by,

$$[J^{\rho\sigma}, J^{\tau\nu}] = \eta^{\sigma\tau} J^{\rho\nu} - \eta^{\rho\tau} J^{\sigma\nu} + \eta^{\tau\nu} J^{\sigma\tau} - \eta^{\sigma\nu} J^{\rho\tau}$$

2.6 Representation of Lie groups

The representation is a linear map from the tangent space of the identity of the manifold to the bounded linear operators of a Hilbert space, $\mathcal{H}$.

$$\pi : T_{\mathbb{I}}\mathcal{M} \to \mathcal{B}(\mathcal{H})$$

In which the following property of Lie bracket is observed,

$$[\pi(x), \pi(y)] = \pi([x, y]) \tag{2.9}$$

We got a representation of $\mathcal{M}$ via exponentiation:

$$\pi\big(g = e^{sX}\big) = e^{\left(e^{s\pi(x)}\right)} \tag{2.10}$$

Therefore, we can focus on this Lie algebra of the group, which is a linear space and try and find the matrix representation of the Lie algebra such that they follow the property given by equation (2.9). Using equation (2.10), we can then, directly get the representations of the group.

2.7 Dirac Equation

We, now try to find the finite dimensional representation of the Lorentz group, using a method devised by Dirac. Suppose we had a set of four n×n matrices, called γ^μ, that satisfies the following anti-commutation relation,

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \times \mathbb{I}_{n\times n} \tag{2.11}$$

Here, the anti-commutator brackets $\{\cdot, \cdot\}$ are referred to as the Dirac algebra. Using this, we can write the n-dimensional representation of the Lorentz algebra (or the Lie algebra) as follows,

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \tag{2.12}$$

For the case of the three-dimensional Euclidean metric, the anti-commutator of the Dirac matrices becomes,

$$\{\gamma^i, \gamma^j\} = i^2\big(\sigma^i\sigma^j + \sigma^j\sigma^i\big) = -2\delta^{ij}$$

where the Dirac matrices have been defined using the Pauli matrices σ^j, as follows,

$$\gamma^j = i\sigma^j$$

So, the matrix representation of the Lorentz algebra (two-dimensional representation) of the rotation group is given by,

$$S^{ij} = \frac{1}{2}\epsilon^{ijk}\sigma^k$$

where, ϵ^{ijk} is the Levi-Civita symbol.
For the case of four-dimensional Minkowski spacetime, there are four 4×4 unitary matrix representations of the Dirac algebra. The matrix representations are,

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{2.13}$$

The matrix representations in these 2×2 forms is called Weyl representation or Chiral representation. The generator of boost under this representation is given by,

$$\begin{aligned} S^{0i} &= \frac{i}{4}\left[\gamma^0, \gamma^i\right] \\ &= \frac{i}{4}\left[\gamma^0\gamma^i - \gamma^i\gamma^0\right] \\ &= \frac{i}{4}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \\ &= \frac{-i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \qquad (\equiv J^{0i} \text{ found in Lie algebra}) \end{aligned} \tag{2.14}$$

The generator of rotation in this representation is given by,

$$\begin{aligned} S^{ij} &= \frac{i}{4}\left[\gamma^i, \gamma^j\right] \\ &= \frac{i}{4}\left[\gamma^i\gamma^j - \gamma^j\gamma^i\right] \\ &= \frac{i}{4}\left[\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\right] \\ &= \frac{-i}{4}\begin{pmatrix} \left[\sigma^i, \sigma^j\right] & 0 \\ 0 & \left[\sigma^i, \sigma^j\right] \end{pmatrix} \end{aligned}$$

Now, the commutation of the Pauli matrices is given by the relation,

$$\left[\sigma^i, \sigma^j\right] = 2\epsilon^{ijk}\sigma^k$$

Therefore, the generator of rotation becomes,

$$S^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \qquad (\equiv J^{\rho\sigma} \text{ foind in Lie algebra}) \tag{2.15}$$

A four-component field ψ that shows boosts and rotation transformation like equation (2.14) and equation (2.15), is called Dirac spinor. Here, the boost generators S^{0i} are not Hermitian and thus, boost implementation is non-unitary. In fact, since Lorentz group has an unbound manifold/non-compact topology, it will have no faithful, finite-dimensional representations that are unitary. But since ψ is considered to be a classical field, it is not an issue.
The transformation of the Dirac spinor is as follows,

$$\psi \to e^{\frac{-i}{2}\omega_{\mu\nu}S^{\mu\nu}}\psi$$

The equation (2.14) and equation (2.15) represent the transformation laws for ψ.
The Klein-Gordon equation is also appropriate, in the sense that the transformation equations (2.14) and (2.15) operate only in the internal space and moves out of the $\left(\partial^2 + m^2\right)$ term. But from an another property of γ matrices, we can write another first order equation that implies to Klein-Gordon equation as well as with more information. Now, consider the commutator,

$$[\gamma^\mu, S^{\rho\sigma}] = (J^{\rho\sigma})^\mu{}_\nu \gamma^\nu \tag{2.16}$$

The above can be equivalently be written as,

$$\left(1 + \frac{i}{2}\omega_{\rho\sigma} S^{\rho\sigma}\right)\gamma^\mu \left(1 - \frac{i}{2}\omega_{\rho\sigma} S^{\rho\sigma}\right) = \left(1 - \frac{i}{2}\omega_{\rho\sigma} J^{\rho\sigma}\right)^\mu{}_\nu \gamma^\nu \tag{2.17}$$

If we consider the above transformation to be infinitesimal and let,

$$\Lambda_{1/2} = \exp\left(-\frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}\right)$$

Where, $\Lambda_{1/2}$ is the Lorentz transformation for Dirac spinors. Also, we consider the following,

$$\Lambda^\mu{}_\nu = \exp\left(-\frac{i}{2}\omega_{\rho\sigma} J^{\rho\sigma}\right)^\mu{}_\nu$$

Where, $\Lambda^\mu{}_\nu$ is the ordinary Lorentz transformation. Then, equation (2.17) can be written as,

$$\Lambda^{-1}{}_{1/2}\gamma^\mu \Lambda_{1/2} = \Lambda^\mu{}_\nu \gamma^\nu \tag{2.18}$$

The Lorentz transformation of four-γ matrices transform like the Lorentz transformation of a 4-vector. Here, $\Lambda_{1/2}$ is called the spinor representation of the Lorentz transformation, Λ. From equation (2.18), we can say that the γ matrices are invariant under simultaneous rotations of their vector and spinor indices, and thus, $\gamma^\mu\partial_\mu$ is also invariant. Keeping the above transformation properties in mind, the Dirac equation is given by,

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \tag{2.19}$$

where, in the Dirac equation, $(i\gamma^\mu\partial_\mu - m)\psi = 0$, $i\gamma^\mu\partial_\mu - m$ are 4×4 matrices of partial derivative operators.

Chapter 3

Schwarzschild solution to the Einstein Field Equations

We will now be migrating to describing the curved spacetime, which will be used to determine the acceleration of the electron. Here, we consider the electron to be moving in a spherically symmetric and static field. This field is best represented by the Schwarzschild solution to the Einstein field equations, and this solution is a useful approximation that describes the spacetime around many planets and stars, including the Earth and the Sun. This chapter will consist of describing in detail, the mathematics involved in the Schwarzschild solution, including the determination the nature of the spacetime interval, the calculation of the Christoffel symbols, and obtain the Schwarzschild metric using the Einstein field equation. A later part of the chapter will involve the conditions and approximations that are to be imposed for the electron in the spherically symmetric and static field, and modify the metric accordingly.

3.1 Weak Equivalence principle and Strong Equivalence principle

The Weak Equivalence principle states that “all objects must fall at the same rate in a uniform gravitational field, and is independent of its structure and composition.”

The Strong Equivalence principle states that “the laws of Physics must reduce to those of special relativity inside a local freely falling frame of reference.”

3.2 Spherically symmetric and static field

Here, we will consider the gravitational field surrounding a spherically symmetric mass distribution that is at rest. This gives a gravitational field that is spherically symmetric. We will also require that the gravitational field to be static, that is, it is to be invariant under a time reversal (time independent and time symmetric).

For the case of spherical symmetry, we will use spherical polar coordinates t, r, θ, ϕ. In the start, we will consider r to be the parameter that identifies different spherical surfaces that are concentric with the origin. As $r \to \infty$,the space becomes flat and the increments in the values of t and r should be equal increments in true time and distance respectively.

We will also include the rectangular Cartesian coordinates t, x, y, z that are related to the spherical polar coordinates t, r, θ, ϕ in the usual way. This is to find the restrictions that the static nature and spherical symmetry brings to the spacetime interval. In particular, the static character does not allow the terms $\mathrm{d}x\mathrm{d}t, \mathrm{d}y\mathrm{d}t, \mathrm{d}z\mathrm{d}t$ in the spacetime interval, since a time-reversal would imply $t \to -t$ and thus these terms will switch signs, and hence it is the case of non-static. The remaining possible terms are then $\mathrm{d}t^2$ and $\mathrm{d}x^i\mathrm{d}y^j$, with $i, j \in (1, 2, 3)$.

If we consider spherical symmetry, the terms in the spacetime interval must be such that they remain invariant under rotations. From $x, y, z, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z$, the combinations that satisfy this condition are,

$$\begin{aligned} &(x^2 + y^2 + z^2)^{\frac{1}{2}} \\ &\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \\ &x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z \end{aligned} \tag{3.1}$$

or their functions. The polar spherical coordinates and the Cartesian coordinates are related as follows,

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned} \tag{3.2}$$

Thus, the first expression in equation (3.1) becomes,

$$\begin{aligned} \therefore \left(x^2 + y^2 + z^2\right)^{\frac{1}{2}} &= \left[r^2 \sin^2\theta \cos^2\phi + r^2 \sin^2\theta \sin^2\phi + r^2 \cos^2\theta\right]^{\frac{1}{2}} \\ &= r \end{aligned}$$

The second expression in equation (3.1) is the same as the differential line element length, $\mathrm{d}l^2$. In the spherical polar coordinates, the differential line element length is given by,

$$\mathrm{d}l^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\phi^2$$

Now, to obtain the last expression in equation (3.1), let us consider the following expression,

$$r^2 = x^2 + y^2 + z^2 \tag{3.3}$$

Differentiating the above expression, we get,

$$\begin{aligned} 2r\mathrm{d}r &= 2x\mathrm{d}x + 2y\mathrm{d}y + 2z\mathrm{d}z \\ \implies r\mathrm{d}r &= x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z \end{aligned} \tag{3.4}$$

Thus, equation (3.1) becomes,

$$\begin{gathered} r \\ \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\phi^2 \\ r\mathrm{d}r \end{gathered} \tag{3.5}$$

Here, the first and the third expressions in the above, contain only r and since a rotation does not change the value of r, those two expressions are invariant under a rotation transformation. The second expression represents the square of the differential length of a line element, which should remain invariant under a rotation. Hence, the expressions in equation (3.5) and in turn expression (3.1) are invariant under rotation transformations.

The spacetime interval can be considered to be of the form,

$$\mathrm{d}s^2 = X(r)\mathrm{d}t^2 - Y(r)\big(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\big) - Z(r)\mathrm{d}r^2 \tag{3.6}$$

with $X(r)$, $Y(r)$ and $Z(r)$ being functions of r only, that need to be determined. The reason why we only consider them to be functions of r is because θ and ϕ are unambiguous and measuring these coordinates depends on how we can divide the circumference concentric with the origin into equal parts, and this does not require any prior knowledge of the metric. However, in the case of the radial coordinate r, it is necessary to know how it corresponds to the distance measurement, and is ultimately needed for the metric. As a result, we need to find its correlation with distance and hence, r will be used as a parameter. The functions, $X(r), Y(r)$ and $Z(r)$ can be determined using Einstein equations. For this, the equation (3.6) can be further simplified by defining a new radial coordinate,

$$r' = r\sqrt{Y(r)}; \theta' = \theta; \phi' = \phi; t' = t$$

Hence, the spacetime interval becomes,

$$\begin{aligned}\mathrm{d}s^2 &= X'(r')\,\mathrm{d}t'^2 - Y(r)\mathrm{d}r^2 - Y(r)r^2\mathrm{d}\theta^2 - Y(r)r^2\sin^2\theta\mathrm{d}\phi^2 - Z(r)\mathrm{d}r^2\\ &= X'(r')\,\mathrm{d}t'^2 - Y'(r')\,\mathrm{d}r'^2 - r'^2\,\mathrm{d}\theta'^2 - r'^2\sin^2\theta'\,\mathrm{d}\phi'^2\end{aligned} \tag{3.7}$$

The second step in the above equation is obtained after combining all the terms involving $\mathrm{d}r'^2$. It is to be noted that the functions X' and Y' can be written in terms of X, Y and Z.

Furthermore, the unknown functions can be written as exponential functions. Removing the primes in the equation (3.7), we can write the following,

$$\mathrm{d}s^2 = \mathrm{e}^{P(r)}\mathrm{d}t^2 - \mathrm{e}^{Q(r)}\mathrm{d}r^2 - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\mathrm{d}\phi^2 \tag{3.8}$$

In terms of metric tensor, the spacetime interval is written as,

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \tag{3.9}$$

Comparing equations (3.8) and (3.9), we get the metric tensor to be,

$$g_{\mu\nu} = \begin{pmatrix} \mathrm{e}^{P(r)} & 0 & 0 & 0\\ 0 & -\mathrm{e}^{Q(r)} & 0 & 0\\ 0 & 0 & -r^2 & 0\\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix} \tag{3.10}$$

The corresponding metric tensor with the indices raised is as follows,

$$g^{\mu\nu} = \begin{pmatrix} \mathrm{e}^{-P(r)} & 0 & 0 & 0\\ 0 & -\mathrm{e}^{-Q(r)} & 0 & 0\\ 0 & 0 & -\frac{1}{r^2} & 0\\ 0 & 0 & 0 & -\frac{1}{r^2\sin^2\theta} \end{pmatrix} \tag{3.11}$$

The unknown functions are evaluated using the Einstein Field equations. In order to do that, we first have to evaluate the Christoffel symbols, which are defined as follows,

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2}g^{\mu\lambda}\left[\frac{\partial g_{\alpha\lambda}}{\partial x^\beta} + \frac{\partial g_{\beta\lambda}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\lambda}\right] \tag{3.12}$$

with the following property being satisfied,

$$\Gamma^\mu{}_{\alpha\beta} = \Gamma^\mu{}_{\beta\alpha}$$

For ${\Gamma^0}_{00}$,

$$
\begin{aligned}
{\Gamma^0}_{00} &= \frac{1}{2}g^{0\lambda}\left[\frac{\partial g_{0\lambda}}{\partial x^0} + \frac{\partial g_{0\lambda}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\lambda}\right] \\
&= \frac{1}{2}g^{0\lambda}\left[0 + 0 - \frac{\partial\left(\mathrm{e}^{Q(r)}\right)}{\partial x^\lambda}\right] \\
&= \frac{1}{2}g^{00}\left[-\frac{\partial\left(\mathrm{e}^{Q(r)}\right)}{\partial x^0}\right] \\
&= \frac{1}{2}\left(\mathrm{e}^{-Q}\right)[0] \\
\therefore {\Gamma^0}_{00} &= 0
\end{aligned}
$$

For ${\Gamma^0}_{01}$,

$$
\begin{aligned}
{\Gamma^0}_{01} = {\Gamma^0}_{10} &= \frac{1}{2}g^{0\lambda}\left[\frac{\partial g_{0\lambda}}{\partial x^1} + \frac{\partial g_{1\lambda}}{\partial x^0} - \frac{\partial g_{10}}{\partial x^\lambda}\right] \\
&= \frac{1}{2}\mathrm{e}^{-Q}\left[\mathrm{e}^{Q}\frac{\partial Q}{\partial r} + 0 - 0\right] = 0 \\
\therefore {\Gamma^0}_{01} = {\Gamma^0}_{10} &= \frac{1}{2}\frac{\partial Q}{\partial r}
\end{aligned}
$$

For ${\Gamma^0}_{02}$,

$$
\begin{aligned}
{\Gamma^0}_{02} = {\Gamma^0}_{20} &= \frac{1}{2}g^{0\lambda}\left[\frac{\partial g_{0\lambda}}{\partial x^2} + \frac{\partial g_{2\lambda}}{\partial x^0} - \frac{\partial g_{20}}{\partial x^\lambda}\right] \\
&= \frac{1}{2}g^{00}\left[\frac{\partial \mathrm{e}^{Q}}{\partial \theta} + \frac{\partial g_{20}}{\partial t} - 0\right] \\
\therefore {\Gamma^0}_{02} = {\Gamma^0}_{20} &= 0
\end{aligned}
$$

For ${\Gamma^0}_{03}$,

$$
\begin{aligned}
{\Gamma^0}_{03} = {\Gamma^0}_{30} &= \frac{1}{2}g^{0\lambda}\left[\frac{\partial g_{0\lambda}}{\partial x^3} + \frac{\partial g_{3\lambda}}{\partial x^0} - \frac{\partial g_{30}}{\partial x^\lambda}\right] \\
&= \frac{1}{2}g^{00}\left[\frac{\partial \mathrm{e}^{Q}}{\partial \phi} + \frac{\partial g_{30}}{\partial t} - 0\right] \\
\therefore {\Gamma^0}_{03} = {\Gamma^0}_{30} &= 0
\end{aligned}
$$

For ${\Gamma^0}_{ij}$,

$$
\begin{aligned}
{\Gamma^0}_{ij} &= \frac{1}{2}g^{0\lambda}\left[\frac{\partial g_{i\lambda}}{\partial x^j} + \frac{\partial g_{j\lambda}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\lambda}\right] \\
&= \frac{1}{2}g^{00}\left[\frac{\partial g_{i0}}{\partial x^j} + \frac{\partial g_{j0}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^0}\right]
\end{aligned}
$$

If $i \neq j$, then $g_{ij} = g^{ij} = 0$.

$$\therefore {\Gamma^0}_{ij} = 0 \text{ (only if } i \neq j)$$

If $i = j$, then,

$$\begin{aligned}\Gamma^0{}_{ii} &= \frac{1}{2}\mathrm{e}^{-Q}\left[2\frac{\partial g_{i0}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^0}\right] \\ &= \frac{1}{2}\mathrm{e}^{-Q}\left[0 - \frac{\partial g_{ii}}{\partial t}\right] = 0\end{aligned}$$

Thus, we can write, in general,

$$\Gamma^0{}_{ij} = 0$$

Now, for $\Gamma^1{}_{00}$,

$$\begin{aligned}\Gamma^1{}_{00} &= \frac{g^{1\lambda}}{2}\left[\frac{\partial g_{0\lambda}}{\partial x^0} + \frac{\partial g_{\lambda 0}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\lambda}\right] \\ &= \frac{g^{11}}{2}\left[0 + 0 - \frac{\partial \mathrm{e}^Q}{\partial r}\right] \\ &= \frac{\mathrm{e}^{-P}}{2}\left[\mathrm{e}^N\frac{\partial Q}{\partial r}\right] \\ \therefore \Gamma^1{}_{00} &= \frac{\mathrm{e}^{Q-P}}{2}\frac{\partial Q}{\partial r}\end{aligned}$$

For $\Gamma^1{}_{0i}$,

$$\begin{aligned}\Gamma^1{}_{0i} = \Gamma^1{}_{i0} &= \frac{g^{1\lambda}}{2}\left[\frac{\partial g_{0\lambda}}{\partial x^i} + \frac{\partial g_{i\lambda}}{\partial x^0} - \frac{\partial g_{0i}}{\partial x^\lambda}\right] \\ &= \frac{g^{11}}{2}\left[\frac{\partial g_{01}}{\partial x^i} + \frac{\partial g_{i1}}{\partial x^0} - 0\right] \\ &= -\frac{\mathrm{e}^{-P}}{2}\left[\frac{\partial g_{i1}}{\partial x^0}\right]\end{aligned}$$

If $i = 1$, then,

$$\Gamma^1{}_{01} = \Gamma^1{}_{10} = -\frac{\mathrm{e}^{-P}}{2}\left[\frac{\partial g_{11}}{\partial t}\right] = 0$$

If $i \neq 1$, then,

$$\Gamma^1{}_{0i} = \Gamma^1{}_{i0} = -\frac{\mathrm{e}^{-P}}{2}\left[\frac{\partial g_{1i}}{\partial t}\right] = 0$$

$$\therefore \Gamma^1{}_{10} = \Gamma^1{}_{20} = \Gamma^1{}_{30} = \Gamma^1{}_{01} = \Gamma^1{}_{02} = \Gamma^1{}_{03} = 0$$

Now, for $\Gamma^1{}_{ij}$,

$$\begin{aligned}\Gamma^1{}_{ij} &= \frac{g^{1\lambda}}{2}\left[\frac{\partial g_{i\lambda}}{\partial x^j} + \frac{\partial g_{j\lambda}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\lambda}\right] \\ &= \frac{g^{11}}{2}\left[\frac{\partial g_{i1}}{\partial x^j} + \frac{\partial g_{j1}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^1}\right] \\ &= -\frac{\mathrm{e}^{-P}}{2}\left[\frac{\partial g_{i1}}{\partial x^j} + \frac{\partial g_{j1}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^1}\right]\end{aligned}$$

If $i \neq j$, then $g_{ij} = g^{ij} = 0$, and this implies,

$$\Gamma^1{}_{ij} = -\frac{\mathrm{e}^{-P}}{2}\left[\frac{\partial g_{i1}}{\partial x^j} + \frac{\partial g_{j1}}{\partial x^i}\right]$$

For $i = 1$ and $j \neq 1$ (or vice-versa), we have $\frac{\partial g_{i1}}{\partial x^i} = 0$. Then,

$$\Gamma^1{}_{1j} = \Gamma^1{}_{j1} = -\frac{\mathrm{e}^{-P}}{2}\left[\frac{\partial g_{11}}{\partial x^j}\right] = 0$$

Now, for $i \neq 1$ and $j \neq 1$, then $\frac{\partial g_{i1}}{\partial x^j} = \frac{\partial g_{j1}}{\partial x^i} = 0$, and,

$$\therefore \Gamma^1{}_{ij} = 0 \text{ (for } i \neq 1 \text{ and } j \neq 1)$$

Now, for $i = j$, we have,

$$\Gamma^1{}_{ii} = -\frac{\mathrm{e}^{-P}}{2}\left[2\frac{\partial g_{i1}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^1}\right]$$

$$\therefore \Gamma^1{}_{11} = -\frac{\mathrm{e}^{-P}}{2}\left[2\frac{\partial g_{11}}{\partial r} - \frac{\partial g_{11}}{\partial r}\right]$$
$$= -\frac{\mathrm{e}^{-P}}{2}\left[-\frac{\partial \mathrm{e}^{P}}{\partial r}\right]$$
$$\therefore \Gamma^1{}_{11} = \frac{1}{2}\frac{\partial P}{\partial r}$$

$$\Gamma^1{}_{22} = -\frac{\mathrm{e}^{-P}}{2}\left[2\frac{\partial g_{21}}{\partial \theta} - \frac{\partial g_{22}}{\partial r}\right]$$
$$= -\frac{\mathrm{e}^{-P}}{2}\left[-\frac{\partial\left(-r^2\right)}{\partial r}\right]$$
$$\therefore \Gamma^1{}_{22} = -r\mathrm{e}^{-P}$$

Now, for $\Gamma^1{}_{33}$,

$$\Gamma^1{}_{33} = -\frac{\mathrm{e}^{-P}}{2}\left[2\frac{\partial g_{31}}{\partial \phi} - \frac{\partial g_{33}}{\partial r}\right]$$
$$= \frac{\mathrm{e}^{-P}}{2}\left[\frac{\partial\left(-r^2\sin^2\theta\right)}{\partial r}\right]$$
$$\therefore \Gamma^1{}_{33} = -r\sin^2\theta\mathrm{e}^{-P}$$

For $\Gamma^2{}_{00}$,

$$\Gamma^2{}_{00} = \frac{g^{2\lambda}}{2}\left[\frac{\partial g_{0\lambda}}{\partial x^0} + \frac{\partial g_{\lambda 0}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\lambda}\right]$$
$$= \frac{g^{22}}{2}\left[\frac{\partial g_{02}}{\partial t} + \frac{\partial g_{20}}{\partial t} - \frac{\partial \mathrm{e}^{Q}}{\partial \theta}\right]$$
$$= \frac{g^{22}}{2}[0 + 0 - 0]$$
$$\therefore \Gamma^2{}_{00} = 0$$

For the case of $\Gamma^2{}_{i0} = \Gamma^2{}_{0i}$,

$$\Gamma^2{}_{i0} = \frac{g^{2\lambda}}{2}\left[\frac{\partial g_{i\lambda}}{\partial x^0} + \frac{\partial g_{\lambda 0}}{\partial x^i} - \frac{\partial g_{0i}}{\partial x^\lambda}\right]$$
$$= \frac{g^{22}}{2}\left[\frac{\partial g_{i2}}{\partial x^0} + \frac{\partial g_{20}}{\partial x^i} - \frac{\partial g_{0i}}{\partial x^2}\right]$$
$$= \frac{g^{22}}{2}\left[\frac{\partial g_{i2}}{\partial x^0}\right]$$

For $i = 1, 2, 3$, $\frac{\partial g_{i2}}{\partial x^0} = 0$.

$$\therefore {\Gamma^2}_{10} = {\Gamma^2}_{20} = {\Gamma^2}_{30} = {\Gamma^2}_{01} = {\Gamma^2}_{02} = {\Gamma^2}_{03} = 0$$

For ${\Gamma^2}_{ij}$,

$$\begin{aligned}
{\Gamma^2}_{ij} &= \frac{g^{2\lambda}}{2}\left[\frac{\partial g_{i\lambda}}{\partial x^j} + \frac{\partial g_{\lambda i}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^\lambda}\right] \\
&= \frac{g^{22}}{2}\left[\frac{\partial g_{i2}}{\partial x^j} + \frac{\partial g_{2j}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^2}\right]
\end{aligned}$$

If $i = j$, we have,

$${\Gamma^2}_{ii} = -\frac{r^{-2}}{2}\left[2\frac{\partial g_{i2}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^2}\right]$$

If $i = 1$,

$$\begin{aligned}
{\Gamma^2}_{11} &= -\frac{r^{-2}}{2}\left[2\frac{\partial g_{12}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^2}\right] \\
&= -\frac{r^{-2}}{2}[0 - 0] \\
\therefore {\Gamma^2}_{11} &= 0
\end{aligned}$$

If $i = 2$,

$$\begin{aligned}
{\Gamma^2}_{22} &= -\frac{r^{-2}}{2}\left[2\frac{\partial g_{22}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^2}\right] \\
&= -\frac{r^{-2}}{2}\left[\frac{\partial\left(-r^2\right)}{\partial\theta}\right] \\
\therefore {\Gamma^2}_{22} &= 0
\end{aligned}$$

If $i = 3$,

$$\begin{aligned}
{\Gamma^2}_{33} &= -\frac{r^{-2}}{2}\left[2\frac{\partial g_{32}}{\partial x^3} - \frac{\partial g_{33}}{\partial x^2}\right] \\
&= \frac{r^{-2}}{2}\left[\frac{\partial\left(-r^2\sin^2\theta\right)}{\partial\theta}\right] \\
&= -\frac{1}{2}[2\sin\theta\cos\theta] \\
\therefore {\Gamma^2}_{33} &= -\sin\theta\cos\theta
\end{aligned}$$

In case of $i \neq j$, we have $\frac{\partial g_{ij}}{\partial x^2} = 0$, since for any values of i and j satisfying the mentioned condition, we are considering the non-diagonal components of g_{ij} which is always 0.

$$\therefore {\Gamma^2}_{ij} = \frac{g^{22}}{2}\left[\frac{\partial g_{2i}}{\partial x^j} + \frac{\partial g_{2j}}{\partial x^i}\right]$$

If $i = 1$ and $j = 3$ or vice-versa, then $g_{13} = g_{23} = 0$. Then,

$$\therefore {\Gamma^2}_{13} = {\Gamma^2}_{23} = 0$$

If $i = 2$ and $j = 1$ or vice-versa, then,

$$\begin{aligned}
{\Gamma^2}_{21} = {\Gamma^2}_{12} &= \frac{g^{22}}{2}\left[\frac{\partial g_{21}}{\partial x^2} + \frac{\partial g_{22}}{\partial x^1}\right] \\
&= -\frac{1}{2r^2}\left[\frac{\partial(-r^2)}{\partial r}\right] \\
\therefore {\Gamma^2}_{21} = {\Gamma^2}_{12} &= \frac{1}{r}
\end{aligned}$$

If $i = 2$ and $j = 3$ or vice-versa, then,

$$\begin{aligned}
{\Gamma^2}_{23} = {\Gamma^2}_{32} &= \frac{g^{22}}{2}\left[\frac{\partial g_{21}}{\partial x^2} + \frac{\partial g_{22}}{\partial x^1}\right] \\
&= -\frac{1}{2r^2}\left[\frac{\partial(-r^2)}{\partial \theta}\right] \\
\therefore {\Gamma^2}_{23} = {\Gamma^2}_{32} &= 0
\end{aligned}$$

Now, for ${\Gamma^3}_{00}$,

$$\begin{aligned}
{\Gamma^3}_{00} &= \frac{g^{\lambda 3}}{2}\left[\frac{\partial g_{\lambda 0}}{\partial x^0} + \frac{\partial g_{0\lambda}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\lambda}\right] \\
&= \frac{g^{33}}{2}\left[\frac{\partial g_{30}}{\partial x^0} + \frac{\partial g_{03}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^3}\right] \\
&= -\frac{1}{2r^2\sin^2\theta}\left[0 + 0 - \frac{\partial \mathrm{r}^N}{\partial \phi}\right] \\
\therefore {\Gamma^3}_{00} &= 0
\end{aligned}$$

For the case of ${\Gamma^3}_{0i}$,

$${\Gamma^3}_{0i} = \frac{g^{\lambda 3}}{2}\left[\frac{\partial g_{\lambda i}}{\partial x^0} + \frac{\partial g_{0\lambda}}{\partial x^i} - \frac{\partial g_{0i}}{\partial x^\lambda}\right]$$

But for $i = 1, 2, 3$, $\frac{\partial g_{0i}}{\partial x^\lambda} = 0$, and hence,

$$\begin{aligned}
{\Gamma^3}_{0i} &= \frac{g^{33}}{2}\left[\frac{\partial g_{3i}}{\partial x^0} + \frac{\partial g_{03}}{\partial x^i}\right] \\
&= -\frac{1}{2r^2\sin^2\theta}\left[0 + \frac{\partial g_{3i}}{\partial t}\right] \\
\therefore {\Gamma^3}_{0i} &= 0 \\
\implies {\Gamma^3}_{01} = {\Gamma^3}_{02} &= {\Gamma^3}_{03} = {\Gamma^3}_{10} = {\Gamma^3}_{20} = {\Gamma^3}_{30} = 0
\end{aligned}$$

For the case of ${\Gamma^3}_{ij}$,

$$\begin{aligned}
{\Gamma^3}_{ij} &= \frac{g^{\lambda 3}}{2}\left[\frac{\partial g_{\lambda i}}{\partial x^j} + \frac{\partial g_{\lambda j}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\lambda}\right] \\
&= \frac{g^{33}}{2}\left[\frac{\partial g_{3i}}{\partial x^j} + \frac{\partial g_{3j}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^3}\right]
\end{aligned}$$

Let us consider the case where $i \neq j$. If $i = 1$, $j = 2$ or vice-versa, we have $g_{3i} = g_{3j} = 0$. Also, $g_{ij} = 0$. In that case,

$${\Gamma^3}_{12} = {\Gamma^3}_{21} = 0$$

Now, let $i = 3 \implies j = 1, 2$. As a result, $g_{3j} = 0$, and $\frac{\partial g_{33}}{\partial x^j} = \frac{\partial\left(-r^2 \sin^2\theta\right)}{\partial x^j}$.

$$\begin{aligned}\implies \frac{\partial g_{33}}{\partial x^1} &= \frac{\partial\left(-r^2 \sin^2\theta\right)}{\partial r} = -2r\sin^2\theta \\ \frac{\partial g_{33}}{\partial x^2} &= \frac{\partial\left(-r^2 \sin^2\theta\right)}{\partial \theta} = -2r^2\sin\theta\cos\theta\end{aligned}$$

$$\begin{aligned}\therefore \Gamma^3{}_{ij} &= \frac{\partial g^{33}}{\partial 2}\left[\frac{\partial g_{33}}{\partial x^j}\right] \\ \Gamma^3{}_{13} = \Gamma^3{}_{31} &= \frac{g^{33}}{2}\left[\frac{\partial g_{33}}{\partial r}\right] = \frac{1}{2r^2\sin^2\theta}\left[2r\sin^2\theta\right] = \frac{1}{r} \\ \Gamma^3{}_{23} = \Gamma^3{}_{32} &= \frac{g^{33}}{2}\left[\frac{\partial g_{33}}{\partial \theta}\right] = \frac{1}{2r^2\sin^2\theta}\left[2r^2\sin\theta\cos\theta\right] = \cot\theta\end{aligned}$$

Now for $i = j$,

$$\Gamma^3{}_{ii} = \frac{g^{33}}{2}\left[2\frac{\partial g_{3i}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^3}\right]$$

But, $\frac{\partial g_{ii}}{\partial x^3} = \frac{\partial g_{ii}}{\partial \phi} = 0$, since none of the elements in the metric tensor depend on ϕ. Now, $g_{31} = g_{32} = 0$. In case of $i = 3$, we get,

$$\frac{\partial g_{33}}{\partial x^3} = \frac{\partial\left(-r^2 \sin^2\theta\right)}{\partial \phi} = 0$$

$$\begin{aligned}&\therefore \frac{\partial g_{3i}}{\partial x^i} = 0 \text{ (for } i = 1, 2, 3) \\ &\implies \Gamma^3{}_{ii} = 0 \\ &\therefore \Gamma^3{}_{11} = \Gamma^3{}_{22} = \Gamma^3{}_{33} = 0\end{aligned}$$

3.3 Evaluation of the Ricci tensor and Ricci scalars

Now, the Riemann tensor is given by,

$$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\beta\mu,\nu} + \Gamma^\sigma{}_{\beta\nu}\Gamma^\alpha{}_{\sigma\mu} - \Gamma^\sigma{}_{\beta\mu}\Gamma^\alpha{}_{\sigma\nu} \tag{3.13}$$

The Ricci tensor is obtained by the contraction of the Riemann tensor as follows,

$$R_{\beta\mu} \equiv R^\alpha{}_{\beta\mu\alpha} \tag{3.14}$$

Thus, using equation (3.13), we can define Ricci tensor in terms of Christoffel symbols as,

$$R_{\beta\mu} = -\Gamma^\alpha{}_{\beta\mu,\alpha} + \Gamma^\alpha{}_{\beta\alpha,\mu} + \Gamma^\sigma{}_{\beta\alpha}\Gamma^\alpha{}_{\sigma\mu} - \Gamma^\sigma{}_{\beta\mu}\Gamma^\alpha{}_{\sigma\alpha} \tag{3.15}$$

The elements of the Ricci tensor will be evaluated now, using the Christoffel symbols we had previously evaluated. Now, R_{00} will be,

$$\begin{aligned}
R_{00} =& -\Gamma^{\alpha}{}_{00,\alpha} + \Gamma^{\alpha}{}_{0\alpha,0} + \Gamma^{\sigma}{}_{0\alpha}\Gamma^{\alpha}{}_{\sigma 0} - \Gamma^{\sigma}{}_{00}\Gamma^{\alpha}{}_{\sigma\alpha} \\
=& -\Gamma^{0}{}_{01,0} + \Gamma^{1}{}_{00,1} + \Gamma^{1}{}_{01}\Gamma^{1}{}_{10} + \Gamma^{0}{}_{01}\Gamma^{1}{}_{00} + \Gamma^{1}{}_{00}\Gamma^{0}{}_{01} - \Gamma^{1}{}_{00}\left(\Gamma^{0}{}_{10} + \Gamma^{1}{}_{11} + \Gamma^{2}{}_{12} + \Gamma^{3}{}_{13}\right) \\
=& -\frac{\partial}{\partial r}\left(\frac{1}{2}\frac{\partial Q}{\partial r}\mathrm{e}^{(Q-P)}\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\frac{\partial Q}{\partial r}\right) + \frac{1}{2}Q'\frac{1}{2}\frac{\partial Q}{\partial r}\mathrm{e}^{(Q-P)} + \frac{1}{2}\frac{\partial Q}{\partial r}\mathrm{e}^{(Q-P)}\frac{1}{2}\frac{\partial Q}{\partial r} \\
& - \frac{1}{2}\frac{\partial Q}{\partial r}\mathrm{e}^{(Q-P)}\left[\frac{1}{2}\frac{\partial Q}{\partial r} + \frac{2}{r} + \frac{1}{2}\frac{\partial P}{\partial r}\right] \\
=& -\frac{1}{2}\left[\frac{\partial^2 Q}{\partial r^2}\mathrm{e}^{(Q-P)}\right] - \frac{1}{2}\frac{\partial Q}{\partial r}\mathrm{e}^{(Q-P)}\left[\frac{\partial Q}{\partial r} - \frac{\partial P}{\partial r}\right] - 0 + \frac{1}{4}\left(\frac{\partial Q}{\partial r}\right)^2\mathrm{e}^{(Q-P)} - \frac{1}{r}\frac{\partial Q}{\partial r}\mathrm{e}^{(Q-P)} \\
& - \frac{1}{4}\frac{\partial Q}{\partial r}\frac{\partial P}{\partial r}\mathrm{e}^{(Q-P)} \\
\therefore R_{00} =& \mathrm{e}^{(Q-P)}\left[-\frac{1}{2}\frac{\partial^2 Q}{\partial r^2} - \frac{1}{4}\left(\frac{\partial Q}{\partial r}\right)^2 + \frac{1}{4}\left(\frac{\partial Q}{\partial r}\right)\left(\frac{\partial P}{\partial r}\right) - \frac{1}{r}\frac{\partial Q}{\partial r}\right]
\end{aligned}$$

For the case of R_{0i},

$$\begin{aligned}
R_{0i} =& -\Gamma^{\alpha}{}_{0i,\alpha} + \Gamma^{\alpha}{}_{0\alpha,i} + \Gamma^{\sigma}{}_{0\alpha}\Gamma^{\alpha}{}_{\sigma i} - \Gamma^{\sigma}{}_{0i}\Gamma^{\alpha}{}_{\sigma\alpha} \\
=& -\Gamma^{0}{}_{0i,0} + \Gamma^{\alpha}{}_{0\alpha,i} + \Gamma^{\sigma}{}_{00}\Gamma^{0}{}_{\sigma i} + \Gamma^{\sigma}{}_{01}\Gamma^{1}{}_{\sigma i} - \Gamma^{\sigma}{}_{0i}\Gamma^{\alpha}{}_{\sigma\alpha} \\
=& -\frac{\partial}{\partial t}\left(\Gamma^{0}{}_{0i}\right) + \Gamma^{\alpha}{}_{0\alpha,i} + \Gamma^{1}{}_{00}\Gamma^{0}{}_{1i} + \Gamma^{0}{}_{01}\Gamma^{1}{}_{0i} - \Gamma^{\sigma}{}_{0i}\Gamma^{\alpha}{}_{\sigma\alpha}
\end{aligned}$$

Now, $\frac{\partial}{\partial t}\left(\Gamma^{0}{}_{0i}\right) = \Gamma^{\alpha}{}_{0\alpha,i} = \Gamma^{0}{}_{1i} = \Gamma^{1}{}_{0i} = 0$. Thus, R_{0i} becomes,

$$\therefore R_{0i} = -\Gamma^{\sigma}{}_{0i}\Gamma^{\alpha}{}_{\sigma\alpha}$$

But for $i = 2, 3$, $\Gamma^{\sigma}{}_{0i} = 0$, and thus,

$$\therefore R_{02} = R_{03} = 0$$

Now, for R_{01}, we have,

$$R_{01} = -\Gamma^{\sigma}{}_{01}\Gamma^{\alpha}{}_{\sigma\alpha} = -\Gamma^{0}{}_{01}\Gamma^{\alpha}{}_{0\alpha} = 0$$

Since $\Gamma^{\alpha}{}_{0\alpha} = 0$. As a result,

$$\therefore R_{0i} = 0$$

Now, consider the purely spatial components of the Ricci tensor, R_{ij},

$$R_{ij} = \Gamma^{\alpha}{}_{ij,\alpha} - \Gamma^{\alpha}{}_{i\alpha,j} + \Gamma^{\sigma}{}_{i\alpha}\Gamma^{\alpha}{}_{\sigma j} - \Gamma^{\sigma}{}_{ij}\Gamma^{\alpha}{}_{\sigma\alpha}$$

If $i \neq j$, then the $\Gamma^{\alpha}{}_{ij}$ components that give non-zero values are $\Gamma^{2}{}_{12}$, $\Gamma^{2}{}_{21}$, $\Gamma^{3}{}_{31}$, $\Gamma^{3}{}_{13}$, $\Gamma^{3}{}_{23}$, $\Gamma^{3}{}_{32}$. And hence, for $R_{12} = R_{21}$,

$$\begin{aligned}
R_{12} = R_{21} =& \Gamma^{\alpha}{}_{12,\alpha} - \Gamma^{\alpha}{}_{1\alpha,2} + \Gamma^{\sigma}{}_{1\alpha}\Gamma^{\alpha}{}_{\sigma 2} - \Gamma^{\sigma}{}_{12}\Gamma^{\alpha}{}_{\sigma\alpha} \\
=& \Gamma^{2}{}_{12,2} - \Gamma^{0}{}_{10,2} - \Gamma^{1}{}_{11,2} - \Gamma^{2}{}_{12,2} - \Gamma^{3}{}_{13,2} + \Gamma^{\sigma}{}_{10}\Gamma^{0}{}_{\sigma 2} + \Gamma^{\sigma}{}_{11}\Gamma^{1}{}_{\sigma 2} + \Gamma^{\sigma}{}_{12}\Gamma^{2}{}_{\sigma 2} \\
& + \Gamma^{\sigma}{}_{13}\Gamma^{3}{}_{\sigma 2} - \Gamma^{2}{}_{12}\Gamma^{\sigma}{}_{2\sigma}
\end{aligned}$$

Here, ${\Gamma^2}_{12,2} = {\Gamma^0}_{10,2} = {\Gamma^1}_{11,2} = {\Gamma^2}_{12,2} = {\Gamma^3}_{13,2} = 0$, and the above expression reduces to,

$$R_{12} = R_{21} = {\Gamma^0}_{10}{\Gamma^0}_{02} + {\Gamma^1}_{11}{\Gamma^1}_{12} + {\Gamma^2}_{12}{\Gamma^2}_{22} + {\Gamma^3}_{13}{\Gamma^3}_{32} - {\Gamma^2}_{12}{\Gamma^3}_{23}$$

Furthermore, ${\Gamma^0}_{02} = {\Gamma^1}_{12} = {\Gamma^2}_{22} = 0$. Thus, $R_{12} = R_{21}$ finally becomes,

$$\begin{aligned} R_{12} = R_{21} =& {\Gamma^3}_{13}{\Gamma^3}_{32} - {\Gamma^2}_{12}{\Gamma^3}_{23} \\ =& \frac{\cot\theta}{r} - \frac{\cot\theta}{r} \\ \therefore R_{12} = R_{21} =& 0 \end{aligned}$$

Now, for R_{13},

$$\begin{aligned} R_{13} = R_{31} =& {\Gamma^\alpha}_{13,\alpha} - {\Gamma^\alpha}_{1\alpha,3} + {\Gamma^\sigma}_{1\alpha}{\Gamma^\alpha}_{\sigma 3} - {\Gamma^\sigma}_{13}{\Gamma^\alpha}_{\sigma\alpha} \\ =& {\Gamma^3}_{13,3} - {\Gamma^\alpha}_{1\alpha,3} + {\Gamma^0}_{1\alpha}{\Gamma^\alpha}_{03} + {\Gamma^1}_{1\alpha}{\Gamma^\alpha}_{13} + {\Gamma^2}_{1\alpha}{\Gamma^\alpha}_{23} + {\Gamma^3}_{1\alpha}{\Gamma^\alpha}_{33} - {\Gamma^3}_{13}{\Gamma^\alpha}_{3\alpha} \end{aligned}$$

However, ${\Gamma^3}_{13,3} = {\Gamma^\alpha}_{1\alpha,3} = {\Gamma^\alpha}_{03} = 0$. And as a result,

$$R_{13} = R_{31} = {\Gamma^1}_{13}{\Gamma^3}_{13} + {\Gamma^2}_{13}{\Gamma^3}_{23} + {\Gamma^3}_{11}{\Gamma^1}_{33} + {\Gamma^3}_{12}{\Gamma^2}_{33} - {\Gamma^3}_{13}\left[{\Gamma^3}_{33} + {\Gamma^0}_{30} + {\Gamma^1}_{31} + {\Gamma^2}_{32}\right]$$

Here also, we have, ${\Gamma^1}_{13} = {\Gamma^2}_{13} = {\Gamma^3}_{11} = {\Gamma^3}_{12} = {\Gamma^3}_{33} = {\Gamma^0}_{30} = {\Gamma^1}_{31} = {\Gamma^2}_{32} = 0$. Thus, the above result becomes,

$$\therefore R_{13} = R_{31} = 0$$

Now, for $R_{23} = R_{32}$,

$$\begin{aligned} R_{23} = R_{32} =& {\Gamma^\alpha}_{23,\alpha} - {\Gamma^\alpha}_{2\alpha,3} + {\Gamma^\sigma}_{2\alpha}{\Gamma^\alpha}_{\sigma 3} - {\Gamma^\sigma}_{23}{\Gamma^\alpha}_{\sigma\alpha} \\ =& {\Gamma^3}_{23,3} - {\Gamma^\alpha}_{2\alpha,3} + {\Gamma^\sigma}_{20}{\Gamma^0}_{\sigma 3} + {\Gamma^\sigma}_{21}{\Gamma^1}_{\sigma 3} + {\Gamma^\sigma}_{22}{\Gamma^2}_{\sigma 3} + {\Gamma^\sigma}_{23}{\Gamma^3}_{\sigma 3} - {\Gamma^3}_{23}{\Gamma^\alpha}_{3\alpha} \end{aligned}$$

Now, here, ${\Gamma^3}_{23,3} = {\Gamma^\alpha}_{2\alpha,3} = {\Gamma^\sigma}_{20} = 0$. Therefore, the above expression modifies to,

$$R_{23} = R_{32} = {\Gamma^2}_{21}{\Gamma^1}_{23} + {\Gamma^1}_{22}{\Gamma^2}_{13} + {\Gamma^3}_{23}{\Gamma^3}_{33} - {\Gamma^3}_{23}\left[{\Gamma^0}_{30} + {\Gamma^1}_{31} + {\Gamma^2}_{32} + {\Gamma^3}_{33}\right]$$

Here, also, we have, ${\Gamma^1}_{23} = {\Gamma^2}_{13} = {\Gamma^3}_{33} = {\Gamma^0}_{30} = {\Gamma^1}_{31} = {\Gamma^2}_{32} = {\Gamma^3}_{33} = 0$, and hence, the above expression yields,

$$\therefore R_{23} = R_{32} = 0$$

Thus, in general, if $i \neq j$, we have,

$$R_{ij} = 0$$

Now, if $i = j$, then the Ricci tensor can be written as follows,

$$R_{ii} = -{\Gamma^\alpha}_{ii,\alpha} + {\Gamma^\alpha}_{i\alpha,i} + {\Gamma^\sigma}_{i\alpha}{\Gamma^\alpha}_{\sigma i} - {\Gamma^\sigma}_{ii}{\Gamma^\alpha}_{\sigma\alpha}$$

Let us evaluate for $i = 1, 2, 3$, individually. So, R_{11} becomes,

$$\begin{aligned}
R_{11} =& -\Gamma^{\alpha}{}_{11,\alpha} + \Gamma^{\alpha}{}_{1\alpha,1} + \Gamma^{\sigma}{}_{1\alpha}\Gamma^{\alpha}{}_{\sigma 1} - \Gamma^{\sigma}{}_{11}\Gamma^{\alpha}{}_{\sigma\alpha} \\
=& -\Gamma^{1}{}_{11,1} + \left[\Gamma^{0}{}_{10,1} + \Gamma^{1}{}_{11,1} + \Gamma^{2}{}_{12,1} + \Gamma^{3}{}_{13,1}\right] + \Gamma^{\sigma}{}_{10}\Gamma^{0}{}_{\sigma 1} + \Gamma^{\sigma}{}_{11}\Gamma^{1}{}_{\sigma 1} + \Gamma^{\sigma}{}_{12}\Gamma^{2}{}_{\sigma 1} + \Gamma^{\sigma}{}_{13}\Gamma^{3}{}_{\sigma 1} \\
& -\Gamma^{1}{}_{11}\Gamma^{\alpha}{}_{1\alpha} \\
=& -\frac{\partial}{\partial r}\left[\frac{1}{2}\frac{\partial P}{\partial r}\right] + \frac{\partial}{\partial r}\left[\frac{1}{2}\frac{\partial Q}{\partial r} + \frac{1}{2}\frac{\partial P}{\partial r} + \frac{2}{r}\right] + \Gamma^{0}{}_{10}\Gamma^{0}{}_{01} + \Gamma^{1}{}_{11}\Gamma^{1}{}_{11} + \Gamma^{2}{}_{12}\Gamma^{2}{}_{12} + \Gamma^{3}{}_{13}\Gamma^{3}{}_{13} \\
& -\frac{1}{2}\frac{\partial P}{\partial r}\left[\frac{1}{2}\frac{\partial Q}{\partial r} + \frac{1}{2}\frac{\partial P}{\partial r} + \frac{2}{r}\right] \\
=& -\frac{1}{2}\frac{\partial^2 P}{\partial r^2} + \frac{1}{2}\frac{\partial^2 Q}{\partial r^2} + \frac{1}{2}\frac{\partial^2 P}{\partial r^2} - \frac{2}{r^2} + \frac{1}{4}\left(\frac{\partial Q}{\partial r}\right)^2 + \frac{1}{4}\left(\frac{\partial P}{\partial r}\right)^2 + \frac{1}{r^2} + \frac{1}{r^2} - \frac{1}{4}\left(\frac{\partial P}{\partial r}\right)\left(\frac{\partial Q}{\partial r}\right) \\
& -\frac{1}{4}\left(\frac{\partial P}{\partial r}\right)^2 - \frac{1}{r}\left(\frac{\partial P}{\partial r}\right) \\
\therefore R_{11} =& \frac{1}{2}\left(\frac{\partial^2 Q}{\partial r^2}\right) + \frac{1}{4}\left(\frac{\partial Q}{\partial r}\right)^2 - \frac{1}{4}\left(\frac{\partial P}{\partial r}\right)\left(\frac{\partial Q}{\partial r}\right) - \frac{1}{r}\left(\frac{\partial P}{\partial r}\right)
\end{aligned}$$

Now, for R_{22},

$$\begin{aligned}
R_{22} =& -\Gamma^{\alpha}{}_{22,\alpha} + \Gamma^{\alpha}{}_{2\alpha,2} + \Gamma^{\sigma}{}_{2\alpha}\Gamma^{\alpha}{}_{\sigma 2} - \Gamma^{\sigma}{}_{22}\Gamma^{\alpha}{}_{\sigma\alpha} \\
=& -\Gamma^{1}{}_{22,1} + \Gamma^{3}{}_{23,2} + \Gamma^{\sigma}{}_{20}\Gamma^{0}{}_{\sigma 2} + \Gamma^{\sigma}{}_{21}\Gamma^{1}{}_{\sigma 2} + \Gamma^{\sigma}{}_{22}\Gamma^{2}{}_{\sigma 2} + \Gamma^{\sigma}{}_{23}\Gamma^{3}{}_{\sigma 2} - \Gamma^{\sigma}{}_{22}\Gamma^{\alpha}{}_{\sigma\alpha} \\
=& -\frac{\partial}{\partial r}\left(-re^{(-P)}\right) + \frac{\partial}{\partial\theta}\left(\cot\theta\right) + \Gamma^{2}{}_{21}\Gamma^{1}{}_{22} + \Gamma^{1}{}_{22}\Gamma^{2}{}_{12} + \Gamma^{3}{}_{23}\Gamma^{3}{}_{32} \\
& + re^{(-P)}\left[\frac{1}{2}\frac{\partial Q}{\partial r} + \frac{1}{2}\frac{\partial P}{\partial r} + \frac{2}{r}\right] \text{ (Here, } \Gamma^{\sigma}{}_{20} = 0) \\
=& e^{(-P)} - re^{-P}\frac{\partial P}{\partial r} - \csc^2\theta + \frac{1}{r}\left(-re^{(-P)}\right) + \left(-re^{(-P)}\right)\frac{1}{r} + \cot^2\theta + re^{(-P)}\left[\frac{1}{2}\frac{\partial Q}{\partial r} + \frac{1}{2}\frac{\partial P}{\partial r} + \frac{2}{r}\right] \\
=& e^{(-P)} - re^{(-P)}\frac{\partial P}{\partial r} - 1 - e^{(-P)} - e^{(-P)} + \frac{re^{(-P)}}{2}\frac{\partial Q}{\partial r} + \frac{re^{(-P)}}{2}\frac{\partial P}{\partial r} + 2e^{(-P)} \\
\therefore R_{22} =& e^{(-P)}\left[1 + \frac{r}{2}\left(\frac{\partial Q}{\partial r} - \frac{\partial P}{\partial r}\right)\right] - 1
\end{aligned}$$

For the case of R_{33}, we have,

$$\begin{aligned}
R_{33} =& -\Gamma^{\alpha}{}_{33,\alpha} + \Gamma^{\alpha}{}_{3\alpha,3} + \Gamma^{\sigma}{}_{3\alpha}\Gamma^{\alpha}{}_{\sigma 3} - \Gamma^{\sigma}{}_{33}\Gamma^{\alpha}{}_{\sigma\alpha} \\
=& -\left[\Gamma^{1}{}_{33,1} + \Gamma^{2}{}_{33,2}\right] + \frac{\partial}{\partial\phi}\left(\Gamma^{\alpha}{}_{3\alpha}\right) + \Gamma^{\sigma}{}_{31}\Gamma^{1}{}_{\sigma 3} + \Gamma^{\sigma}{}_{32}\Gamma^{2}{}_{\sigma 3} + \Gamma^{\sigma}{}_{33}\Gamma^{3}{}_{\sigma 3} - \Gamma^{1}{}_{33}\Gamma^{\alpha}{}_{1\alpha} - \Gamma^{2}{}_{33}\Gamma^{\alpha}{}_{2\alpha} \\
=& \left[\frac{\partial}{\partial r}\left(r\sin^2\theta e^{(-P)}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\cos\theta\right)\right] + 0 + \Gamma^{3}{}_{31}\Gamma^{1}{}_{33} + \Gamma^{3}{}_{32}\Gamma^{2}{}_{33} + \Gamma^{1}{}_{33}\Gamma^{3}{}_{13} + \Gamma^{2}{}_{33}\Gamma^{3}{}_{23} \\
& -\Gamma^{1}{}_{33}\left[\Gamma^{0}{}_{10} + \Gamma^{1}{}_{11} + \Gamma^{2}{}_{12} + \Gamma^{3}{}_{13}\right] - \Gamma^{2}{}_{33}\left[\Gamma^{0}{}_{20} + \Gamma^{1}{}_{21} + \Gamma^{2}{}_{22} + \Gamma^{3}{}_{23}\right]
\end{aligned}$$

But, $\Gamma^0{}_{20} = \Gamma^1{}_{21} = \Gamma^2{}_{22} = 0$. And therefore,

$$\begin{aligned}
R_{33} =& \mathrm{e}^{(-P)} \sin^2\theta - \mathrm{e}^{(-P)} r \sin^2\theta \left(\frac{\partial P}{\partial r}\right) + \cos^2\theta - \sin^2\theta - \mathrm{e}^{(-P)} \sin^2\theta - \cos^2\theta - \mathrm{e}^{(-P)} \sin^2\theta \\
& - \cos^2\theta + \cos^2\theta + r\mathrm{e}^{(-P)} \sin^2\theta \left[\frac{1}{2}\frac{\partial Q}{\partial r} + \frac{1}{2}\frac{\partial P}{\partial r} + \frac{2}{r}\right] \\
=& -\frac{\mathrm{e}^{(-P)} r \sin^2\theta}{2}\left(\frac{\partial P}{\partial r}\right) + \mathrm{e}^{(-P)} \sin^2\theta + \frac{\mathrm{e}^{(-P)} r \sin^2\theta}{2}\left(\frac{\partial Q}{\partial r}\right) - \sin^2\theta \\
\therefore R_{33} =& \mathrm{e}^{(-L)} \sin^2\theta \left[1 + \frac{r}{2}\left(\frac{\partial Q}{\partial r} - \frac{\partial P}{\partial r}\right)\right] - \sin^2\theta
\end{aligned}$$

We now evaluate the Ricci scalar, which is determined by contracting the Ricci tensor,

$$\begin{aligned}
R =& R^\alpha{}_\alpha \\
=& R^0{}_0 + R^1{}_1 + R^2{}_2 + R^3{}_3 \\
=& g^{0\alpha} R_{\alpha 0} + g^{1\beta} R_{\beta 1} + g^{2\gamma} R_{\gamma 2} + g^{3\delta} R_{\delta 3} \\
=& g^{00} R_{00} + g^{11} R_{11} + g^{22} R_{22} + g^{33} R_{33} \\
=& \mathrm{e}^{(-Q)} \mathrm{e}^{(Q-P)} \left[-\frac{1}{2}\frac{\partial^2 Q}{\partial r^2} + \frac{1}{4}\left(\frac{\partial P}{\partial r}\right)\left(\frac{\partial Q}{\partial r}\right) - \frac{1}{4}\left(\frac{\partial Q}{\partial r}\right)^2 - \frac{1}{r}\frac{\partial Q}{\partial r}\right] \\
& - \mathrm{e}^{(-P)} \left[\frac{1}{2}\frac{\partial^2 Q}{\partial r^2} - \frac{1}{4}\left(\frac{\partial P}{\partial r}\right)\left(\frac{\partial Q}{\partial r}\right) + \frac{1}{4}\left(\frac{\partial Q}{\partial r}\right)^2 - \frac{1}{r}\left(\frac{\partial P}{\partial r}\right)\right] - \frac{\mathrm{e}^{(-P)}}{r^2}\left[1 + \frac{r}{2}\left(\frac{\partial Q}{\partial r} - \frac{\partial P}{\partial r}\right)\right] \\
& + \frac{1}{r^2} - \frac{1}{r^2 \sin^2\theta}\left[\mathrm{e}^{(-P)} \sin^2\theta \left[1 + \frac{r}{2}\left(\frac{\partial Q}{\partial r} - \frac{\partial P}{\partial r}\right)\right] - \sin^2\theta\right] \\
=& \mathrm{e}^{(-P)} \left[\frac{1}{2}\frac{\partial^2 Q}{\partial r^2} + \frac{1}{4}\left(\frac{\partial P}{\partial r}\right)\left(\frac{\partial Q}{\partial r}\right) - \frac{1}{4}\left(\frac{\partial Q}{\partial r}\right)^2 - \frac{1}{r}\frac{\partial Q}{\partial r} - \frac{1}{2}\frac{\partial^2 N}{\partial r^2} + \frac{1}{4}\left(\frac{\partial P}{\partial r}\right)\left(\frac{\partial Q}{\partial r}\right)\right. \\
& \left. - \frac{1}{4}\left(\frac{\partial Q}{\partial r}\right)^2 + \frac{1}{r}\left(\frac{\partial P}{\partial r}\right) - \frac{1}{r^2} - \frac{1}{2r}\left(\frac{\partial Q}{\partial r} - \frac{\partial P}{\partial r}\right) - \frac{1}{r^2} - \frac{1}{2r}\left(\frac{\partial Q}{\partial r} - \frac{\partial P}{\partial r}\right)\right] + \frac{2}{r^2} \\
\therefore R =& \mathrm{e}^{(-P)} \left[-\frac{\partial^2 Q}{\partial r^2} + \frac{1}{2}\left(\frac{\partial P}{\partial r}\right)\left(\frac{\partial Q}{\partial r}\right) - \frac{1}{2}\left(\frac{\partial Q}{\partial r}\right)^2 + \frac{2}{r}\left(\frac{\partial P}{\partial r} - \frac{\partial Q}{\partial r}\right) - \frac{2}{r^2}\right] + \frac{2}{r^2}
\end{aligned}$$

3.4 Solutions of the Einstein Field equation

Now, the Einstein field equation is,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{3.16}$$

In case of vacuum, the Energy-Momentum tensor, $T_{\mu\nu}$ becomes zero. The Einstein field equation, then, becomes,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \tag{3.17}$$

Multiplying both sides by the metric tensor $g^{\nu\rho}$, the above equation further modifies to,

$$\begin{aligned}
R_{\mu\nu} g^{\nu\rho} - \frac{1}{2} g_{\mu\nu} g^{\nu\rho} R = 0 \\
\therefore R_\mu{}^\rho - \frac{1}{2} \delta_\mu{}^\rho R = 0
\end{aligned}$$

Thus, we can individually, write the field equations as,

$$
\begin{aligned}
& R_0{}^0 - \frac{1}{2}R = 0 \\
\implies & R_{0\alpha}g^{0\alpha} - \frac{R}{2} = 0 \\
\implies & R_{00}g^{00} - \frac{R}{2} = 0 \\
\implies & \mathrm{e}^{(Q-P)}\left[-\frac{1}{2}\frac{\partial^2 Q}{\partial r^2} - \frac{1}{4}\left(\frac{\partial Q}{\partial r}\right)^2 + \frac{1}{4}\left(\frac{\partial Q}{\partial r}\right)\left(\frac{\partial P}{\partial r}\right) - \frac{1}{r}\left(\frac{\partial Q}{\partial r}\right)\right]\mathrm{e}^{(-Q)} \\
& - \frac{\mathrm{e}^{(-P)}}{2}\left[-\frac{\partial^2 Q}{\partial r^2} + \frac{1}{2}\left(\frac{\partial P}{\partial r}\right)\left(\frac{\partial Q}{\partial r}\right) - \frac{1}{2}\left(\frac{\partial Q}{\partial r}\right)^2 + \frac{2}{r}\left(\frac{\partial P}{\partial r} - \frac{\partial Q}{\partial r}\right) - \frac{2}{r^2}\right] - \frac{1}{r^2} = 0 \\
\implies & \mathrm{e}^{(-P)}\Bigg[- \frac{1}{2}\frac{\partial^2 Q}{\partial r^2} - \frac{1}{4}\left(\frac{\partial Q}{\partial r}\right)^2 + \frac{1}{4}\left(\frac{\partial Q}{\partial r}\right)\left(\frac{\partial P}{\partial r}\right) - \frac{1}{r}\left(\frac{\partial Q}{\partial r}\right) + \frac{1}{2}\frac{\partial^2 Q}{\partial r^2} - \frac{1}{4}\left(\frac{\partial Q}{\partial r}\right)\left(\frac{\partial P}{\partial r}\right) \\
& + \frac{1}{4}\left(\frac{\partial Q}{\partial r}\right)^2 - \frac{1}{r}\left(\frac{\partial P}{\partial r} - \frac{\partial P}{\partial r}\right) + \frac{1}{r^2}\Bigg] - \frac{1}{r^2} = 0 \\
\implies & \frac{\mathrm{e}^{(-P)}}{r}\left[\frac{1}{r} - \frac{\partial P}{\partial r}\right] - \frac{1}{r^2} = 0
\end{aligned} \tag{3.18}
$$

$$
\begin{aligned}
& R_1{}^1 - \frac{1}{2}R = 0 \\
\implies & R_{1\alpha}g^{\alpha 1} - \frac{R}{2} = 0 \\
\implies & R_{11}g^{11} - \frac{R}{2} = 0 \\
\implies & -\left[\frac{1}{2}\left(\frac{\partial^2 Q}{\partial r^2}\right) - \frac{1}{4}\left(\frac{\partial P}{\partial r}\right)\left(\frac{\partial Q}{\partial r}\right) + \frac{1}{4}\left(\frac{\partial Q}{\partial r}\right)^2 - \frac{1}{r}\frac{\partial P}{\partial r}\right]\mathrm{e}^{(-P)} - \frac{\mathrm{e}^{(-P)}}{2}\Bigg[-\frac{\partial^2 Q}{\partial r^2} \\
& + \frac{1}{2}\left(\frac{\partial P}{\partial r}\right)\left(\frac{\partial Q}{\partial r}\right) - \frac{1}{2}\left(\frac{\partial Q}{\partial r}\right)^2 + \frac{2}{r}\left(\frac{\partial P}{\partial r} - \frac{\partial Q}{\partial r}\right) - \frac{2}{r^2}\Bigg] - \frac{1}{r^2} = 0 \\
\implies & \mathrm{e}^{(-P)}\Bigg[\frac{1}{2}\left(\frac{\partial^2 Q}{\partial r^2}\right) - \frac{1}{4}\left(\frac{\partial P}{\partial r}\right)\left(\frac{\partial Q}{\partial r}\right) + \frac{1}{4}\left(\frac{\partial Q}{\partial r}\right)^2 - \frac{1}{r}\left(\frac{\partial P}{\partial r}\right) - \frac{1}{2}\left(\frac{\partial^2 Q}{\partial r^2}\right) + \frac{1}{4}\left(\frac{\partial P}{\partial r}\right)\left(\frac{\partial Q}{\partial r}\right) \\
& - \frac{1}{4}\left(\frac{\partial Q}{\partial r}\right)^2 + \frac{1}{r}\left(\frac{\partial P}{\partial r} - \frac{\partial Q}{\partial r}\right) - \frac{1}{r^2}\Bigg] - \frac{1}{r^2} = 0 \\
\implies & \mathrm{e}^{(-P)}\left[\frac{1}{r}\left(\frac{\partial Q}{\partial r}\right) + \frac{1}{r^2}\right] - \frac{1}{r^2} = 0
\end{aligned} \tag{3.19}
$$

$$
\begin{aligned}
&R_2{}^2 - \frac{1}{2}\delta_2{}^2 R = 0\\
\implies &R_{2\alpha} g^{\alpha 2} - \frac{R}{2} = 0\\
\implies &R_{22} g^{22} - \frac{R}{2} = 0\\
\implies &\frac{\mathrm{e}^{(-P)}}{r^2}\left[1 + \frac{r}{2}\left(\frac{\partial Q}{\partial r} - \frac{\partial P}{\partial r}\right)\right] + \frac{1}{r^2} - \frac{\mathrm{e}^{(-P)}}{2}\left[-\frac{\partial^2 Q}{\partial r^2} + \frac{1}{2}\left(\frac{\partial P}{\partial r}\right)\left(\frac{\partial Q}{\partial r}\right) - \frac{1}{2}\left(\frac{\partial Q}{\partial r}\right)^2\right.\\
&\left. + \frac{2}{r}\left(\frac{\partial P}{\partial r} - \frac{\partial Q}{\partial r}\right) - \frac{2}{r^2}\right] - \frac{1}{r^2} = 0\\
\implies &\mathrm{e}^{(-P)}\left[-\frac{1}{r^2} - \frac{1}{2r}\left(\frac{\partial Q}{\partial r} - \frac{\partial P}{\partial r}\right) + \frac{1}{2}\left(\frac{\partial^2 Q}{\partial r^2}\right) - \frac{1}{4}\left(\frac{\partial Q}{\partial r}\right)\left(\frac{\partial P}{\partial r}\right) + \frac{1}{4}\left(\frac{\partial Q}{\partial r}\right)^2 - \frac{1}{r}\left(\frac{\partial P}{\partial r} - \frac{\partial Q}{\partial r}\right)\right.\\
&\left. + \frac{1}{r^2}\right] - \frac{1}{r^2} + \frac{1}{r^2} = 0\\
\implies &\mathrm{e}^{(-P)}\left[\frac{1}{2}\left(\frac{\partial^2 Q}{\partial r^2}\right) - \frac{1}{4}\left(\frac{\partial P}{\partial r}\right)\left(\frac{\partial Q}{\partial r}\right) + \frac{1}{4}\left(\frac{\partial Q}{\partial r}\right)^2 + \frac{1}{2r}\left(\frac{\partial Q}{\partial r} - \frac{\partial P}{\partial r}\right)\right] = 0 \qquad (3.20)
\end{aligned}
$$

$$
\begin{aligned}
&R_3{}^3 - \frac{1}{2}\delta_3{}^3 R = 0\\
\implies &R_{3\alpha} g^{\alpha 3} - \frac{R}{2} = 0\\
\implies &R_{33} g^{33} - \frac{R}{2} = 0\\
\implies &\mathrm{e}^{(-P)}\sin^2\theta\left(\frac{-1}{r^2\sin^2\theta}\right)\left[1 + \frac{r}{2}\left(\frac{\partial Q}{\partial r} - \frac{\partial P}{\partial r}\right)\right] + \frac{\sin^2\theta}{r^2\sin^2\theta} - \frac{\mathrm{e}^{(-P)}}{2}\left[-\frac{\partial^2 Q}{\partial r^2} + \frac{1}{2}\left(\frac{\partial P}{\partial r}\right)\left(\frac{\partial Q}{\partial r}\right)\right.\\
&\left. - \frac{1}{2}\left(\frac{\partial Q}{\partial r}\right)^2 + \frac{2}{r}\left(\frac{\partial P}{\partial r} - \frac{\partial Q}{\partial r}\right) - \frac{2}{r^2}\right] - \frac{1}{r^2} = 0\\
\implies &-\mathrm{e}^{(-P)}\left[\frac{1}{r^2} + \frac{1}{2r}\left(\frac{\partial Q}{\partial r} - \frac{\partial P}{\partial r}\right) - \frac{1}{2}\frac{\partial^2 Q}{\partial r^2} + \frac{1}{4}\left(\frac{\partial P}{\partial r}\right)\left(\frac{\partial Q}{\partial r}\right) - \frac{1}{4}\left(\frac{\partial Q}{\partial r}\right)^2 + \frac{1}{r}\left(\frac{\partial P}{\partial r} - \frac{\partial Q}{\partial r}\right)\right.\\
&\left. - \frac{1}{r^2}\right] = 0\\
\implies &\mathrm{e}^{(-P)}\left[\frac{1}{2}\frac{\partial^2 Q}{\partial r^2} + \frac{1}{2r}\left(\frac{\partial Q}{\partial r} - \frac{\partial P}{\partial r}\right) - \frac{1}{4}\left(\frac{\partial Q}{\partial r}\right)\left(\frac{\partial P}{\partial r}\right) + \frac{1}{4}\left(\frac{\partial Q}{\partial r}\right)^2\right] = 0 \qquad (3.21)
\end{aligned}
$$

Consider the equation (3.18),

$$
\frac{\mathrm{e}^{(-P)}}{r}\left[\frac{1}{r} - \frac{\partial P}{\partial r}\right] - \frac{1}{r^2} = 0
$$

This equation can be solved to obtain an expression for P, as follows,

$$\frac{\mathrm{e}^{(-P)}}{r}\left[\frac{1}{r}-\frac{\partial P}{\partial r}\right]-\frac{1}{r^2}=0$$

$$\begin{aligned}
&\implies \mathrm{e}^{(-P)}\left[1-r\frac{\partial P}{\partial r}\right]=1\\
&\implies \left[1-r\frac{\mathrm{d}P}{\mathrm{d}r}\right]=\mathrm{e}^{P}\\
&\implies \mathrm{d}r-r\mathrm{d}P=\mathrm{e}^{P}\mathrm{d}r\\
&\implies -r\mathrm{d}P=\left(\mathrm{e}^{P}-1\right)\mathrm{d}r\\
&\implies -\frac{\mathrm{d}P}{\mathrm{e}^{P}-1}=\frac{\mathrm{d}r}{r}
\end{aligned}$$

Let $x=\mathrm{e}^{P}-1$. Then,

$$\begin{aligned}
&\mathrm{e}^{P}\mathrm{d}P=\mathrm{d}x\\
\implies &\mathrm{d}P=\frac{\mathrm{d}x}{1+x}
\end{aligned}$$

Thus, the above differential equation becomes,

$$-\frac{1}{x}\left(\frac{\mathrm{d}x}{1+x}\right)=\frac{\mathrm{d}r}{r}$$

Using partial fraction properties, the left-hand side of the above equation can be written as,

$$\begin{aligned}
\frac{1}{x(x+1)}&=\frac{A}{x}+\frac{B}{x+1}\\
\therefore 1&=A(x+1)+Bx
\end{aligned}$$

- Substituting $x=0$, we get $A=1$.
- Substituting $x=(-1)$, we get $B=(-1)$.

$$\therefore \frac{1}{x(x+1)}=\frac{1}{x}-\frac{1}{x+1}$$

And the differential equation becomes,

$$\begin{aligned}
-\left(\frac{1}{x}-\frac{1}{x+1}\right)\mathrm{d}x&=\frac{\mathrm{d}r}{r}\\
\therefore \left(\frac{1}{x+1}-\frac{1}{x}\right)\mathrm{d}x&=\frac{\mathrm{d}r}{r}
\end{aligned}$$

Integrating both sides, we obtain,

$$\begin{aligned}
\ln(1+x) - \ln(x) &= \ln(r) + C \\
\therefore \ln\left(\frac{1+x}{x}\right) &= \ln(Cr) \\
\therefore \frac{1+\mathrm{e}^P - 1}{\mathrm{e}^P - 1} &= Cr \\
\therefore \frac{\mathrm{e}^P}{\mathrm{e}^P - 1} &= Cr \\
\therefore 1 - \mathrm{e}^{(-P)} &= \frac{C}{r} \\
\therefore \mathrm{e}^P &= \frac{1}{1 - \frac{C}{r}}
\end{aligned}$$

In the above set of equations, C is the constant of integration. Now, we will consider the result obtained in equation (3.19),

$$\mathrm{e}^{(-L)}\left[\frac{1}{r}\left(\frac{\partial N}{\partial r}\right) + \frac{1}{r^2}\right] - \frac{1}{r^2} = 0$$

When we subtract equation (3.19) from (3.18), we obtain the following,

$$\begin{aligned}
&\mathrm{e}^{(-P)}\left[\frac{1}{r^2} - \frac{1}{r}\left(\frac{\partial P}{\partial r}\right)\right] - \frac{1}{r^2} - \mathrm{e}^{(-P)}\left[\frac{1}{r}\left(\frac{\partial Q}{\partial r}\right) + \frac{1}{r^2}\right] + \frac{1}{r^2} = 0 \\
\Longrightarrow\ &\mathrm{e}^{(-P)}\left[\frac{1}{r^2} - \frac{1}{r}\left(\frac{\partial P}{\partial r}\right) - \frac{1}{r}\left(\frac{\partial Q}{\partial r}\right) - \frac{1}{r^2}\right] = 0 \\
\Longrightarrow\ &\frac{1}{r}\left(\frac{\partial P}{\partial r}\right) = -\frac{1}{r}\left(\frac{\partial Q}{\partial r}\right) \\
\Longrightarrow\ &\frac{\partial P}{\partial r} = -\frac{\partial Q}{\partial r} \\
\Longrightarrow\ &P = -Q + C_1
\end{aligned}$$

where, C is again, the constant of integration. Now at very large distances ($r \to \infty$), the metric tensor for spherically symmetric and static field reduces to the flat spacetime metric in spherical coordinates. Taking limits on both sides of the above expression,

$$\lim_{r\to\infty} P = -\lim_{r\to\infty} Q + C_1 \tag{3.22}$$

But,

$$\lim_{r\to\infty}\begin{pmatrix} \mathrm{e}^Q & 0 & 0 & 0 \\ 0 & -\mathrm{e}^P & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}$$

Thus, from equation (3.22), we can write,

$$\begin{aligned}
\lim_{r\to\infty} \mathrm{e}^Q = 1 &\Longrightarrow \lim_{r\to\infty} Q = 0 \\
\lim_{r\to\infty} -\mathrm{e}^P = -1 &\Longrightarrow \lim_{r\to\infty} P = 0
\end{aligned}$$

Substituting these two results in equation (3.22), we obtain,

$$
\begin{aligned}
&0 = -0 + C_1 \\
\implies &C_1 = 0 \\
\implies &P = -Q \\
\implies &\mathrm{e}^{Q} = \mathrm{e}^{-P} = 1 - \frac{C}{r}
\end{aligned}
$$

$$
\therefore \mathrm{e}^{Q} = \mathrm{e}^{-P} = 1 - \frac{C}{r} \tag{3.23}
$$

Therefore, the spacetime interval for spherically symmetric static field is given by,

$$
\mathrm{d}s^2 = \left(1 - \frac{C}{r}\right)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{\left(1 - \frac{C}{r}\right)} - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\mathrm{d}\phi^2 \tag{3.24}
$$

The value of C can be obtained by comparing the above equation with the metric tensor obtained using linear theory, which is derived using the equations of a spherically symmetric, static field. In that, it is found that $\frac{C}{r}$ is proportional to the Newtonian gravitational potential. Using linear theory, we obtain the spacetime interval to be,

$$
\mathrm{d}s^2 \approx \left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 - \left(1 + \frac{2GM}{r}\right)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) \tag{3.25}
$$

where, M is the central mass. Here, the equations (3.24) and (3.25) cannot be compared directly since the radial variable r, is defined quite differently in the two cases. In equation (3.24), r is defined such that the circumference of the circle that is concentric with the origin is $2\pi r$, whereas, the r defined in equation (3.25) is such that the circumference of of a similar circle will be $2\pi r\left(1 + \frac{GM}{r}\right)$.

To make the two equations consistent with each other, we modify equation (3.25) by introducing another new coordinate, given by,

$$
\begin{aligned}
&r' = \frac{1}{2}\sqrt{r^2 - Cr} + \frac{r}{2} - \frac{C}{4} \\
\implies &4r' = 2\sqrt{r^2 - Cr} + 2r - C \\
\implies &4r' - 2r + C = 2\sqrt{r^2 - Cr} \\
\implies &\left(4r' - 2r + C\right)^2 = 4\left(r^2 - Cr\right) \\
\implies &16(r')^2 + 4r^2 + C^2 - 16r'r + 8Cr' - 4Cr = 4r^2 - 4Cr \\
\implies &16r'r = 16(r')^2 + 8Cr' + C^2 \\
\implies &16r'r = \left(4r' + C\right)^2 \\
\implies &r = \frac{(4r')^2}{16r'}\left(1 + \frac{C}{4r'}\right)^2 \\
\implies &r = r'\left(1 + \frac{C}{4r'}\right)^2
\end{aligned}
$$

Differentiating both sides of the above expression, we get,

$$\begin{aligned}
\mathrm{d}r =&\mathrm{d}r'\left(1+\frac{C}{4r'}\right)^2 + 2r'\left(1+\frac{C}{4r'}\right)\left(\frac{-C}{4(r')^2}\right)\mathrm{d}r' \\
=&\left(1+\frac{C}{4r'}\right)\mathrm{d}r'\left[\left(1+\frac{C}{4r'}\right)-\frac{C}{2r'}\right] \\
=&\left(1+\frac{C}{4r'}\right)\left(1-\frac{C}{4r'}\right)\mathrm{d}r'
\end{aligned} \tag{3.26}$$

Also,

$$\begin{aligned}
\left(1-\frac{C}{r}\right) =&\left[1-\frac{C}{r'}\left(1+\frac{C}{4r'}\right)^{-2}\right] \\
=&\left(1+\frac{C}{4r'}\right)^{-2}\left[\left(1+\frac{C}{4r'}\right)^2-\frac{C}{r'}\right] \\
=&\left(1+\frac{C}{4r'}\right)^{-2}\left[1+\left(\frac{C}{4r'}\right)^2+\frac{C}{2r'}-\frac{C}{r'}\right] \\
=&\left(1+\frac{C}{4r'}\right)^{-2}\left[1-\frac{C}{2r'}+\left(\frac{C}{4r'}\right)^2\right] \\
\therefore \left(1-\frac{C}{r}\right) =&\left(\frac{1-C/4r'}{1+C/4r'}\right)^2
\end{aligned} \tag{3.27}$$

Using the above relations, equation (3.24) can be written as,

$$\begin{aligned}
\mathrm{d}s^2 =&\left(1-\frac{C}{r}\right)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{(1-\frac{C}{r})} - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\mathrm{d}\phi^2 \\
=&\left(\frac{1-C/4r'}{1+C/4r'}\right)^2\mathrm{d}t^2 - \left(\frac{1+C/4r'}{1-C/4r'}\right)^2\left(1+\frac{C}{4r'}\right)^2\left(1-\frac{C}{4r'}\right)^2\mathrm{d}r'^2 \\
&-\left(1+\frac{C}{4r'}\right)^4(r')^2\mathrm{d}\theta^2 - \left(1+\frac{C}{4r'}\right)^4(r')^2\sin^2\theta\mathrm{d}\phi^2 \\
=&\left(\frac{1-C/4r'}{1+C/4r'}\right)^2\mathrm{d}t^2 - \left(1+\frac{C}{4r'}\right)^4\mathrm{d}r'^2 - \left(1+\frac{C}{4r'}\right)^4(r')^2\mathrm{d}\theta^2 - \left(1+\frac{C}{4r'}\right)^4(r')^2\sin^2\theta\mathrm{d}\phi^2 \\
\therefore \mathrm{d}s^2 =&\left(\frac{1-C/4r'}{1+C/4r'}\right)^2\mathrm{d}t^2 - \left(1+\frac{C}{4r'}\right)^4\left[(\mathrm{d}r')^2+(r')^2\mathrm{d}\theta^2+(r')^2\sin^2\theta\mathrm{d}\phi^2\right]
\end{aligned} \tag{3.28}$$

If we consider the case of weak field limit, then $r' \to \infty$ and the coefficients of $\mathrm{d}t^2$ and $(\mathrm{d}r')^2 + (r')^2\mathrm{d}\theta^2 + (r')^2\sin^2\theta\mathrm{d}\phi^2$ will modify as follows,

$$\left(\frac{1-C/4r'}{1+C/4r'}\right)^2 = \left(1-\frac{C}{4r'}\right)^2\left(1+\frac{C}{4r'}\right)^{-2}$$

Let us consider the general expansion of $\left(1+\frac{C}{4r'}\right)^n$,

$$\left(1+\frac{C}{4r'}\right)^n = 1+\frac{nC}{4r'}+\frac{n(n-1)C^2}{32(r')^2}+\ldots$$

Now, substituting $n=-2$ and assuming $r' \to \infty$, we get,

$$\left(1+\frac{C}{4r'}\right)^{-2} \simeq 1-\frac{2C}{4r'} \equiv 1-\frac{C}{2r'}$$

$$
\begin{aligned}
\therefore \left(\frac{1-C/4r'}{1+C/4r'}\right)^2 &= \left(1-\frac{C}{4r'}\right)^2\left(1+\frac{C}{4r'}\right)^{-2} \\
&\simeq \left(1-\frac{C}{4r'}\right)^2\left(1-\frac{C}{2r'}\right) \\
&\simeq \left(1-\frac{C}{2r'}\right)\left(1-\frac{C}{2r'}\right) \\
&\simeq 1-\frac{C}{2r'}-\frac{C}{2r'}+\frac{C}{4(r')^2} \\
\therefore \left(\frac{1-C/4r'}{1+C/4r'}\right)^2 &\approx \left(1-\frac{C}{r'}\right) \text{ (as } r'\to\infty)
\end{aligned}
$$

And for the case of $\left(1+\frac{C}{4r'}\right)^4$,

$$
\begin{aligned}
\left(1+\frac{C}{4r'}\right)^4 &= \left[\left(1+\frac{C}{4r'}\right)^2\right]^2 \\
&= \left[1+\frac{2C}{4r'}+\frac{C^2}{16(r')^2}\right]^2 \\
&= \left[1+\frac{C^2}{4(r')^2}+\frac{C^4}{256(r')^4}+\frac{C}{r'}+\frac{C^2}{8(r')^2}+\frac{C^3}{16(r')^3}\right]
\end{aligned}
$$

But considering the weak field limit, $r'\to\infty$, the above expression reduces to,

$$
\therefore \left(1+\frac{C}{4r'}\right)^4 \simeq 1+\frac{C}{r'}
$$

It is worthwhile to note that in the weak field limit, we will consider the r' terms only of the order of -1. Ignoring the r' terms of order of -1 will result in a flat spacetime interval, and hence it is important that it should be considered.

Using the above relations, the equation (3.28) reduces to,

$$
\mathrm{d}s^2 \simeq \left[1-\frac{C}{r'}\right]\mathrm{d}t^2 - \left(1+\frac{C}{r'}\right)\left[(\mathrm{d}r')^2+(r')^2\mathrm{d}\theta^2+(r')^2\sin^2\theta\mathrm{d}\phi^2\right] \tag{3.29}
$$

Equation (3.29) is consistent with equation (3.25), and comparing the two equations, we find that,

$$
C = 2GM
$$

and as a result, equation (3.24) becomes,

$$
\mathrm{d}s^2 = \left(1-\frac{2GM}{r}\right)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{\left(1-\frac{2GM}{r}\right)} - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\mathrm{d}\phi^2 \tag{3.30}
$$

This is the Schwarzschild solution. The mass M in the above equations is the total mass of the system; even the mass-energy due to the gravitational fields is to be considered. Using equation (3.30), the metric of a spherically symmetric and static field, also called the Schwarzschild metric can be written as,

$$
g^{\mu\nu} = \begin{pmatrix} \left(1-\frac{2GM}{r}\right) & 0 & 0 & 0 \\ 0 & -\frac{1}{1-2GM/r} & 0 & 0 \\ 0 & 0 & -\frac{1}{1-2GM/r} & 0 \\ 0 & 0 & 0 & -\frac{1}{1-2GM/r} \end{pmatrix} \tag{3.31}
$$

3.5 Schwarzschild metric for the electron system

We will now proceed to obtain the spacetime metric of our electron system. Consider a uniform gravitational field in which the gravitational force $\mathbf{F}_G$, acts on an object of mass m (which in our case, will be the electron). Let the distance between this object and a body of mass M (which could be a star or planet) be r. The gravitational force $\mathbf{F}_G$ can then, be written using Newtonian gravitation as,

$$\mathbf{F}_G = -\frac{GMm}{r^2}\hat{\mathbf{e}}_z \tag{3.32}$$

Here, G is the universal gravitational constant and $\hat{\mathbf{e}}_z$ is the unit vector along the z-axis. Now, let us consider a coordinate system that contains and is in the vicinity of the electron, with its origin located at a distance of R_0 from the body of mass M, as shown in the diagram below,

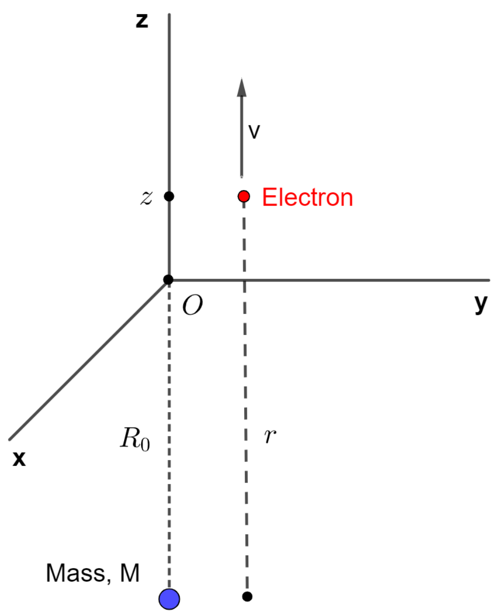

Figure 3.1: Figure representing the motion of the electron and the coordinate frame in which it is located, along with the body of mass M.

Here,

$$r = R_0 + z$$

where, z represents the coordinates of the test mass (electron) in the coordinate system frame, shown in the above diagram. If we consider R_0 (and subsequently r) to be sufficiently large, then equation (3.32) reduces to,

$$
\begin{aligned}
\mathbf{F}_G =& -\frac{GMm}{r^2}\hat{\mathbf{e}}_z \\
=& -\frac{GMm}{(R_0+z)^2}\hat{\mathbf{e}}_z \\
=& -\frac{GMm}{(R_0)^2}\left(1+\frac{z}{R_0}\right)^{-2}\hat{\mathbf{e}}_z \\
=& -\frac{GMm}{(R_0)^2}\left(1-\frac{2z}{R_0}+\dots\right)\hat{\mathbf{e}}_z \\
\therefore \mathbf{F}_G \approx& -\frac{GMm}{(R_0)^2}\hat{\mathbf{e}}_z
\end{aligned}
\tag{3.33}
$$

after we have neglected the terms of the order of $\frac{1}{{R_0}^3}$, which is considered to be negligible in our case. Therefore, we can have the gravitational force $\mathbf{F}_G$ to be approximately independent of the position of the electron. Now, to determine the Schwarzschild metric for the above conditions, let us calculate the term $\left(1-\frac{2GM}{r}\right)^{-1}$ present in equation (3.31),

$$
\left(1-\frac{2GM}{r}\right)^{-1} = 1+\frac{2GM}{R_0+z}+\left(\frac{2GM}{R_0+z}\right)^2+\dots \simeq 1+\frac{2GM}{R_0+z}
$$

after considering only the first-order terms. Thus, the metric in equation (3.31) modifies to,

$$
g^{\mu\nu} = \begin{pmatrix}
\left(1-\frac{2GM}{R_0+z}\right) & 0 & 0 & 0 \\
0 & -\left(1+\frac{2GM}{R_0+z}\right) & 0 & 0 \\
0 & 0 & -\left(1+\frac{2GM}{R_0+z}\right) & 0 \\
0 & 0 & 0 & -\left(1+\frac{2GM}{R_0+z}\right)
\end{pmatrix}
$$

Let $\Phi_G(z) = \frac{GM}{R_0+z}$, and the above metric can be written as,

$$
g^{\mu\nu} = \begin{pmatrix}
(1-2\Phi_G) & 0 & 0 & 0 \\
0 & -(1+2\Phi_G) & 0 & 0 \\
0 & 0 & -(1+2\Phi_G) & 0 \\
0 & 0 & 0 & -(1+2\Phi_G)
\end{pmatrix}
\tag{3.34}
$$

As mentioned before, only the first-order terms of Φ_G was included in the equation (3.34), as it will be done in the rest of this report. The metric represented in equation (3.34) will be used to describe the spacetime in which the electron accelerates. It is also possible to show that $\nabla\Phi_G$ is a constant,

$$
\begin{aligned}
\nabla\Phi_G =& \frac{\partial}{\partial z}\left(\frac{GM}{R_0+z}\right) \\
=& -\frac{GM}{(R_0+z)^2} \\
=& -\frac{GM}{{R_0}^2}\left(1+\frac{z}{R_0}\right)^{-2} \\
=& -\frac{GM}{{R_0}^2}\left(1-\frac{2z}{R_0}+\dots\right) \\
\therefore \nabla\Phi_G \approx& -\frac{GM}{{R_0}^2} \implies \text{constant}
\end{aligned}
$$

We need to find the corresponding Dirac matrices $\gamma^{\mu}(x)$, such that they satisfy the following condition,

$$\gamma^{\mu}(x)\gamma^{\nu}(x) + \gamma^{\nu}(x)\gamma^{\mu}(x) = 2g^{\mu\nu} \times \mathbb{I}$$

Suppose, γ^{α} and ${\gamma_{(0)}}^{\beta}$ are the Dirac matrices in the spherically symmetric, static spacetime and in the flat spacetime respectively. These two Dirac matrices can be related as follows,

$$\gamma^{\alpha} = {V_{\beta}}^{\alpha}(z){\gamma_{(0)}}^{\beta} \tag{3.35}$$

Here, the tetrad $V^{\alpha}_{\ \beta}$ can be used to relate the Schwarzschild metric and the Minkowski flat spacetime metric as follows,

$$g^{\mu\nu} = {V_{\alpha}}^{\mu}{V_{\beta}}^{\nu}\eta^{\alpha\beta}$$

For g^{00}, we have,

$$\begin{aligned} g^{00} &= {V_{\alpha}}^{0}{V_{\beta}}^{0}\eta^{\alpha\beta} \\ &= \left({V_0}^{0}\right)^2 - \left({V_1}^{0}\right)^2 - \left({V_2}^{0}\right)^2 - \left({V_3}^{0}\right)^2 \end{aligned}$$

To a certain extent, we can assume the values of some of the tetrad elements, such that the transformation remains valid. If we take, ${V_1}^{0} = {V_2}^{0} = {V_3}^{0} = 0$, then the above expression becomes,

$$\begin{aligned} g^{00} &= \left({V_0}^{0}\right)^2 \\ \implies {V_0}^{0} &= \sqrt{1 - 2\Phi_G} \\ &= 1 - \frac{2\Phi_G}{2} + \frac{1}{2}\left(-\frac{1}{2}\right)(2\Phi_G)^2 - \dots \\ \therefore {V_0}^{0} &\approx 1 - \Phi_G \end{aligned} \tag{3.36}$$

In the above expression, the higher order terms are ignored since, $|\Phi_G|/c^2 \ll 1$.

In the case of g^{ii}, where $i = 1, 2, 3$,

$$\begin{aligned} g^{ii} &= {V_{\alpha}}^{i}{V_{\beta}}^{i}\eta^{\alpha\beta} \\ &= \left({V_0}^{i}\right)^2 - \left({V_j}^{i}\right)^2 \end{aligned}$$

Here, if we take ${V_0}^{i} = {V_j}^{i} = 0$, for $i \neq j$, then, we have,

$$-\left({V_i}^{i}\right)^2 = g^{ii}$$

But, $g^{ii} = -(1 + 2\Phi_G)$. Thus, the above expression modifies to,

$$\begin{aligned} \left({V_i}^{i}\right)^2 &= 1 + 2\Phi_G \\ \implies {V_i}^{i} &= \sqrt{1 + 2\Phi_G} \\ \therefore {V_i}^{i} &\approx 1 + \Phi_G \end{aligned} \tag{3.37}$$

In the above expression also, the higher order terms have been ignored, since $|\Phi_G|/c^2 \ll 1$. Using equation (3.36) and equation (3.37), equation (3.35) can be written as follows,

$$\gamma^{0}(z) = (1 - \Phi_G(z)){\gamma_{(0)}}^{0} \tag{3.38}$$

$$\gamma^{i}(z) = (1 + \Phi_G(z)){\gamma_{(0)}}^{i} \tag{3.39}$$

Chapter 4

Tetrad Formalism and Dirac Equation in curved spacetime

4.1 Transformation properties of Dirac Equation in flat spacetime

In the flat Minkowski spacetime, the Dirac equation is given by,

$$i{\gamma_{(0)}}^{\mu}\partial_{\mu}\psi(x) = m\psi(x) \tag{4.1}$$

where, m is the mass of the particle.

Here, we will make use of the Planck natural units, in which $c = \hbar = 1$. Also, ${\gamma_{(0)}}^{\mu}$ refers to the Dirac matrices with the subscript (0) indicating that the Dirac matrices used here, refer to the ones used in flat Minkowski spacetime. ${\gamma_{(0)}}^{\mu}$ are required to satisfy the condition,

$$\left\{{\gamma_{(0)}}^{\mu}, {\gamma_{(0)}}^{\nu}\right\} = {\gamma_{(0)}}^{\mu}{\gamma_{(0)}}^{\nu} + {\gamma_{(0)}}^{\nu}{\gamma_{(0)}}^{\mu} = 2\eta^{\mu\nu}\mathbb{I}_{4\times 4} \tag{4.2}$$

with $\{\cdot,\cdot\}$ referring to the anti-commutator brackets of the Dirac algebra, and $\eta^{\mu\nu}$ referring to the Minkowski metric tensor.

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

We will now try to determine how the Dirac equation transforms from one inertial frame to another. For that, we first consider the homogeneous Lorentz transformation from old coordinates 'x^{μ}' to the new coordinates '$x^{\mu'}$', in flat spacetime.

$$x^{\mu'} = \Lambda^{\mu'}{}_{\nu}x^{\nu} \tag{4.3}$$

where, the Lorentz transformation matrix $\Lambda^{\mu'}{}_{\nu}$, reflects the boosts and rotations of spacetime.

Let the quantum system be defined by an m-component wavefunction, Ψ_m (which can be considered to be a field), then its Lorentz transformation can be written as,

$$\Psi'_m = \sum_{n=1}^{m} [S(\Lambda)]_{mn}\Psi_n \tag{4.4}$$

Here, $S(\Lambda)$ is an $m \times m$, irreducible matrix representation of the Lorentz group. The properties of $S(\Lambda)$ can be found by considering an infinitesimal Lorentz transformation,

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu} \tag{4.5}$$

where $\delta^{\mu}{}_{\nu}$ is the Kronecker delta and $\omega^{\mu}{}_{\nu}$ is the infinitesimal parameter, rotations and boosts in the transformation. Thus, the representation for such a transformation can be written as,

$$S(\Lambda) = S(1+\omega) = 1 + \frac{1}{2}\omega^{\alpha\beta}\sigma_{\alpha\beta} \tag{4.6}$$

The above expansion is obtained using Taylor series expansion and $\sigma_{\alpha\beta}$ can be considered to be the basis of the Lie algebra of the Lorentz group, and are 4×4 matrices.

Also, since $S(\Lambda)$ forms the representation of the Lorentz group, the following multiplication law will be satisfied,

$$S(\Lambda_1)S(\Lambda_2) = S(\Lambda_1\Lambda_2) = S(\Lambda_3) \tag{4.7}$$

Now, any homogeneous Lorentz transformation matrix, must satisfy the condition,

$$\Lambda^{\alpha}{}_{\gamma}\eta_{\alpha\beta}\Lambda^{\beta}{}_{\delta} = \eta_{\gamma\delta} \tag{4.8}$$

Substituting $\Lambda^{\alpha}{}_{\gamma} = \delta^{\alpha}{}_{\gamma} + \omega^{\alpha}{}_{\gamma}$ and $\Lambda^{\beta}{}_{\delta} = \delta^{\beta}{}_{\delta} + \omega^{\beta}{}_{\delta}$ into the above equation,

$$\begin{aligned}
\left(\delta^{\alpha}{}_{\gamma} + \omega^{\alpha}{}_{\gamma}\right)\eta_{\alpha\beta}\left(\delta^{\beta}{}_{\delta} + \omega^{\beta}{}_{\delta}\right) &= \eta_{\gamma\delta} \\
\therefore \delta^{\beta}{}_{\delta}\delta^{\alpha}{}_{\gamma}\eta_{\alpha\beta} + \eta_{\alpha\beta}\delta^{\beta}{}_{\delta}\omega^{\alpha}{}_{\gamma} + \delta^{\alpha}{}_{\gamma}\omega_{\alpha\delta} + \omega_{\alpha\delta}\omega^{\alpha}{}_{\gamma} &= \eta_{\gamma\delta} \\
\therefore \omega_{\delta\gamma} + \omega_{\gamma\delta} &= 0 \quad \text{(Upto first order of } \omega) \\
\therefore \omega_{\delta\gamma} &= -\omega_{\gamma\delta}
\end{aligned} \tag{4.9}$$

Now, we have defined the infinitesimal Lorentz transformation as,

$$S(\Lambda) = S(1+\omega) = 1 + \frac{1}{2}\omega^{\alpha\beta}\sigma_{\alpha\beta} \tag{4.10}$$

We can choose the basis, $\sigma_{\alpha\beta}$ to be anti-symmetric in α and β,

$$\sigma_{\alpha\beta} = -\sigma_{\beta\alpha} \tag{4.11}$$

In case of the Lorentz transformation of a contravariant vector, $S(\Lambda)$ is simply the Lorentz boost matrix, $\Lambda^{\alpha'}{}_{\alpha}$,

$$V^{\alpha} \to V^{\alpha'} \equiv \Lambda^{\alpha'}{}_{\alpha}V^{\alpha} \tag{4.12}$$

In case of a rank-2 mixed tensor, the representation , $S(\Lambda)$ in this case is $\Lambda^{\alpha'}{}_{\alpha}\Lambda_{\beta'}{}^{\beta}$,

$$T^{\alpha}{}_{\beta} \to T^{\alpha'}{}_{\beta'} \equiv \Lambda^{\alpha'}{}_{\alpha}\Lambda_{\beta'}{}^{\beta}T^{\alpha}{}_{\beta} \tag{4.13}$$

The problem of finding the representations of infinitesimal homogeneous Lorentz group, can be figured by finding out those matrices satisfying the relation,

$$[\sigma_{\alpha\beta}, \sigma_{\gamma\delta}] = \eta_{\gamma\beta}\omega_{\alpha\delta} - \eta_{\gamma\alpha}\omega_{\beta\delta} + \eta_{\delta\beta}\omega_{\gamma\alpha} - \eta_{\delta\alpha}\omega_{\gamma\beta} \tag{4.14}$$

For the case of Dirac theory, $\sigma_{\alpha\beta}$ are 4×4 matrices, defined by,

$$\begin{aligned}
\sigma^{\mu\nu} &= \frac{1}{4}\left[\gamma_{(0)}{}^{\mu}, \gamma_{(0)}{}^{\nu}\right] \\
&= \frac{1}{4}\left(\gamma_{(0)}{}^{\mu}\gamma_{(0)}{}^{\nu} - \gamma_{(0)}{}^{\nu}\gamma_{(0)}{}^{\mu}\right)
\end{aligned} \tag{4.15}$$

The Dirac equation must be independent of the Lorentz frame, that is, the equation must be invariant under the choice of coordinates.

Let us consider an inhomogeneous Lorentz transformation, $x^{'} = \Lambda x + a$, where Λ is the Lorentz transformation matrix, satisfying $g^{'} = \Lambda^{\intercal} g \Lambda$, (where g is the metric tensor), and 'a' represents a shift in spacetime. The Dirac equation will remain invariant if we define,

$$\begin{aligned}\psi'\big(x'\big) &= S(\Lambda)\psi(x) \\ &= S(\Lambda)\psi\big(\Lambda^{-1}\big(x' - a\big)\big)\end{aligned} \tag{4.16}$$

where, the Lorentz group representation, $S(\Lambda)$, is a 4×4 matrix operating on the components of ψ. The above equation can be written in tensor notation as,

$$\psi'_{\alpha}(x) = S_{\alpha}{}^{\beta}(\Lambda)\psi_{\beta}\big(\Lambda^{-1}(x - a)\big) \tag{4.17}$$

The index β, follows the Einstein summation convention.

To say that the Dirac equation is invariant under a Lorentz transformation, is by saying that in the new frame, ψ' obeys the Dirac equation in the new frame,

$$\Big(-i\gamma^{\nu'}\partial_{\nu'} + m\Big)\psi'\big(x'\big) = 0 \tag{4.18}$$

In the original frame, the Dirac equation is as follows,

$$\big(-i\gamma^{\mu}\partial_{\mu} + m\big)\psi(x) = 0 \tag{4.19}$$

But,

$$\frac{\partial}{\partial x^{\mu}} = \frac{\partial x^{\nu'}}{\partial x^{\mu}}\frac{\partial}{\partial x^{\nu'}} = \Lambda^{\nu'}{}_{\mu}\partial_{\nu'}$$

Using equation (4.16), equation (4.19) becomes,

$$\begin{aligned}&\Big(-i\gamma^{\mu}\Lambda^{\nu'}{}_{\mu}\partial_{\nu'} + m\Big)S^{-1}(\Lambda)\psi'\big(x'\big) = 0 \\ \therefore &\left[-i\gamma^{\mu}\Lambda^{\nu'}{}_{\mu}\frac{\partial}{\partial x^{\nu'}}\Big(S^{-1}\psi'\big(x'\big)\Big)\right] + mS^{-1}\psi'\big(x'\big) = 0\end{aligned}$$

Multiplying both sides of the above equation by S, we get,

$$-\,iS\Big(\Lambda^{\nu'}{}_{\mu}\gamma^{\mu}\Big)S^{-1}\partial_{\nu'}\psi'\big(x'\big) + m\psi'\big(x'\big) = 0 \tag{4.20}$$

But the Dirac equation should be invariant under Lorentz transformation and must retain the form as seen in equation (4.18). Comparing both the equations (4.18) and (4.20), we obtain the condition that has to be imposed on $S(\Lambda)$, the Lorentz group representation,

$$S^{-1}(\Lambda)\gamma^{\nu'}S(\Lambda) = \Lambda^{\nu'}{}_{\mu}\gamma^{\mu} = \gamma^{\nu'} \tag{4.21}$$

The above result suggests that the Dirac matrices are invariant under a Lorentz transformation. From the above equation, we can also obtain the anti-commutation relation between the Dirac matrices in the new primed frame,

$$\begin{aligned}\gamma^{\mu'} &= \Lambda^{\mu'}{}_{\alpha}\gamma^{\alpha} \\ \gamma^{\nu'} &= \Lambda^{\nu'}{}_{\beta}\gamma^{\beta}\end{aligned} \tag{4.22}$$

$$\begin{aligned}
\left\{\gamma^{\mu'}, \gamma^{\nu'}\right\} &= \gamma^{\mu'}\gamma^{\nu'} + \gamma^{\nu'}\gamma^{\mu'} \\
&= \Lambda^{\mu'}{}_{\alpha}\gamma^{\alpha}\Lambda^{\nu'}{}_{\beta}\gamma^{\beta} + \Lambda^{\nu'}{}_{\beta}\gamma^{\beta}\Lambda^{\mu'}{}_{\alpha}\gamma^{\alpha} \\
&= \Lambda^{\mu'}{}_{\alpha}\Lambda^{\nu'}{}_{\beta}\left(\gamma^{\alpha}\gamma^{\beta} + \gamma^{\beta}\gamma^{\alpha}\right) \\
&= \Lambda^{\mu'}{}_{\alpha}\Lambda^{\nu'}{}_{\beta}\left(2g^{\alpha\beta}\right) \\
\therefore \left\{\gamma^{\mu'}, \gamma^{\nu'}\right\} &= 2g^{\mu'\nu'} \qquad (4.23)
\end{aligned}$$

Since this is dealt in flat spacetime, equation (4.18) and equation (4.21) can be written as,

$$S^{-1}(\Lambda)\gamma_{(0)}{}^{\mu}S(\Lambda) = \Lambda^{\mu}{}_{\nu}\gamma_{(0)}{}^{\mu} \qquad (4.24)$$

$$i\gamma_{(0)}{}^{\mu}\partial_{\mu}\psi = m\psi \qquad (4.25)$$

4.2 Tetrad formalism

For a given set of special relativistic equations that can be used to describe a system in flat Minkowski spacetime, that is, in the absence of gravity, then, we replace all the Lorentz tensors in that equation with quantities that act like tensors, after a general coordinate transformation. Under this formalism, we replace all the derivatives ∂_α, with its respective covariant derivative, while the Minkowski metric $\eta_{\alpha\beta}$, which is described for the case of flat spacetime, will be replaced by the general metric tensor $g_{\mu\nu}$. On doing these, the equations of motions are generally of covariant nature.

The above formalism is applicable for quantities that transform like tensors under a Lorentz transformation. This is therefore, not applicable for spinors, since there are no representations of the $GL(4)$ group (general linear group of 4×4 matrices) that behave like spinors under Lorentz subgroup.

In order to incorporate spinors, let us consider a set of coordinates ξ^α at each and every spacetime event, X, such that ξ^α is locally inertial at X. In case of a non-inertial general coordinate system, the metric is given by,

$$g_{\mu\nu}(x) = V^{\alpha}_{\ \mu}(x)V^{\beta}_{\ \nu}(x)\eta_{\alpha\beta} \qquad (4.26)$$

where, $V^{\alpha}_{\ \mu}(x) = \left.\frac{\partial\xi^\alpha(x)}{\partial x^\mu}\right|_{x=X}$ and $V^{\beta}_{\ \nu}(x)$ is similarly defined. Here, x^μ represents the non-inertial coordinates, and when we shift from x^μ to $x^{\mu'}$, $V^{\alpha}_{\ \mu}$ transforms as follows,

$$V^{\alpha}_{\ \mu} \rightarrow V^{\alpha}_{\ \mu'} \equiv \frac{\partial x^\nu}{\partial x^{\mu'}}\left(\frac{\partial\xi^\alpha(x)}{\partial x^\nu}\right)\bigg|_{x=X} \qquad (4.27)$$

We can consider $V^{\alpha}_{\ \nu}$ to consist of 4 covariant vector fields and not as a single tensor. These set of four vectors are known as tetrads or vierbeins. For a given covariant vector field $A^\mu(x)$, we use the tetrad to refer its components at x to the coordinate system ξ^α, that is locally inertial at x,

$$^*T^\alpha = V^{\alpha}_{\ \mu}T^\mu \qquad (4.28)$$

This resembles a contraction of the contravariant vector, A^μ with the four covariant vectors $V^{\alpha}_{\ \mu}$, giving four scalar quantities, represented by $^*A^\alpha$. In case of covariant vectors,

$$^*T_\alpha = V_{\alpha}{}^{\mu}T_\mu \qquad (4.29)$$

and for a general mixed tensor,

$$ {}^{*}T^{\alpha}_{\ \ \beta} = V^{\alpha}_{\ \ \mu} V_{\beta}^{\ \ \nu} T^{\mu}_{\ \ \nu} \tag{4.30} $$

$V_{\alpha}^{\ \ \beta}$ and $V_{\beta}^{\ \ \nu}$ are also both tetrads, with lowering of indices (inertial coordinate indices) done using Minkowski metric and the raising of index (non-inertial coordinate indices) done using the respective metric tensor,

$$ V_{\beta}^{\ \ \nu} = \eta_{\alpha\beta} g^{\mu\nu} V^{\alpha}_{\ \ \mu} \tag{4.31} $$

Now, as per the principle of Equivalence, special relativity is applicable in locally inertial frames, and is independent of the locally inertial frame chosen at that point. Since the scalar field components (like ${}^{*}T^{\alpha}$ and ${}^{*}T_{\alpha\beta}$) are defined using a local inertial coordinate system, they must transform like Lorentz transformation from one frame to another. Hence,

$$ {}^{*}T_{\alpha\beta}(x) \rightarrow \Lambda_{\alpha}^{\ \ \pi}(x) \Lambda_{\beta}^{\ \ \phi}(x) {}^{*}T_{\pi\phi}(x) \tag{4.32} $$

where, $\Lambda_{\alpha}^{\ \ \beta}(x)$ is the spacetime dependent Lorentz transformation. In case of a general field, ${}^{*}\psi_{n}(x)$, the transformation rule is,

$$ {}^{*}\psi_{a}(x) \rightarrow \sum_{m} [D(\Lambda)]_{ab} {}^{*}\psi_{b}(x) \tag{4.33} $$

where, $[D(\Lambda)]_{ab}$ is either the identity, the tensor representation or the spinor representation of the infinitesimal Lorentz group, for transformations on the local inertial coordinate frame. The quantities transforming as per equation (4.33) is called Lorentz scalar, Lorentz tensor or Lorentz spinor, depending on the manner of representation of the Lorentz group.

Under a coordinate transformation, an ordinary derivative transforms like,

$$ \frac{\partial}{\partial x^{\mu}} \rightarrow \frac{\partial}{\partial x^{\mu'}} \equiv \frac{\partial x^{\nu}}{\partial x^{\mu'}} \frac{\partial}{\partial x^{\nu}} \tag{4.34} $$

If the action had fields that were coordinate scalars, there will be no way to contract the covariant index, μ'. In order to do contract that index, we make use of tetrad as follows,

$$ V_{\alpha}^{\ \ \mu} \frac{\partial}{\partial x^{\mu}} $$

This forms four scalars and is a coordinate scalar.

However, if we consider the Lorentz transformation of this quantity acting on a general field ${}^{*}\psi$, we get,

$$ \begin{aligned} V_{\alpha}^{\ \ \mu}(x) \frac{\partial}{\partial x^{\mu}} \left({}^{*}\psi(x) \right) &\rightarrow \Lambda_{\alpha}^{\ \ \beta}(x) V_{\beta}^{\ \ \mu}(x) \frac{\partial}{\partial x^{\mu}} \left[D(\Lambda(x)) {}^{*}\psi(x) \right] \\ &\equiv \Lambda_{\alpha}^{\ \ \beta}(x) V_{\beta}^{\ \ \mu}(x) \left[D(\Lambda(x)) \frac{\partial}{\partial x^{\mu}} \left(\psi(x) \right) + \frac{\partial}{\partial x^{\mu}} \left(D(\Lambda(x)) \right) {}^{*}\psi(x) \right] \end{aligned} \tag{4.35} $$

But this is not a Lorentz vector. Hence, derivatives need to be used in the action equation as an operator $\mathscr{D}_{\alpha}$, such that it is both a coordinate scalar and Lorentz vector. For a position-dependent Lorentz transformation, $\Lambda^{\alpha}_{\ \ \beta}(x)$,

$$ \mathscr{D}_{\alpha} {}^{*}\psi(x) \rightarrow \Lambda_{\alpha}^{\ \ \beta}(x) D(\Lambda(x)) \mathscr{D}_{\alpha} {}^{*}\psi(x) \tag{4.36} $$

If an action depends on fields, ${}^*\psi$ and their derivatives, $\mathscr{D}_\alpha{}^*\psi$, it will be independent under a transformation of its local inertial frame. From equation (4.35), the derivative $\mathscr{D}_\alpha$, that should be both a Lorentz scalar and a Lorentz vector, can be written as,

$$\mathscr{D}_\alpha = V_\alpha{}^\mu \left[\frac{\partial}{\partial x^\mu} + \Gamma_\mu\right] \tag{4.37}$$

The matrix Γ_μ has the following transformation law,

$$\Gamma_\mu \to -\left[\frac{\partial}{\partial x^\mu}\Big(D(\Lambda(x))\Big)\right]D^{-1}(\Lambda(x)) + D(\Lambda(x))\Gamma_\mu(x)D^{-1}(\Lambda(x)) \tag{4.38}$$

Thus, the transformation of $\mathscr{D}_\alpha{}^*\psi$ is as follows,

$$\begin{aligned}
\mathscr{D}_\alpha{}^*\psi &= V_\alpha{}^\mu \left[\frac{\partial}{\partial x^\mu} + \Gamma_\mu\right]{}^*\psi \\
&\to \Lambda_\alpha{}^\beta(x) V_\beta{}^\mu(x)\left[\frac{\partial}{\partial x^\mu} + D(\Lambda(x))\Gamma_\mu(x)D^{-1}(\Lambda(x)) - \left[\frac{\partial}{\partial x^\mu}\Big(D(\Lambda(x))\Big)\right]D^{-1}(\Lambda(x))\right]D(\Lambda)^*\psi \\
&\equiv \Lambda_\alpha{}^\beta(x) V_\beta{}^\mu(x)\left[\left[\frac{\partial}{\partial x^\mu}\Big(D(\Lambda)^*\psi\Big)\right] + \left[D\Big(\Lambda(x)\Big)\Gamma_\mu(x)\right]{}^*\psi(x) - \left[\frac{\partial}{\partial x^\mu}\Big(D(\Lambda)\Big){}^*\psi\right]\right] \\
&\equiv \Lambda_\alpha{}^\beta(x) V_\beta{}^\mu(x)\left[D(\Lambda)\frac{\partial}{\partial x^\mu}\Big({}^*\psi\Big) + D(\Lambda)\Gamma_\mu({}^*\psi)\right]
\end{aligned} \tag{4.39}$$

To determine the structure of $\Gamma_\mu(x)$, let us consider an infinitesimal Lorentz transformation near the identity element given by,

$$\Lambda^\alpha{}_\beta(x) = \delta^\alpha{}_\beta + \omega^\alpha{}_\beta(x)$$

where, $\omega_{\alpha\beta}$ is the infinitesimal parameter that is anti-symmetric,

$$\omega_{\alpha\beta} = -\omega_{\beta\alpha}$$

The representation of this infinitesimal Lorentz transformation is given by,

$$D(1+\omega(x)) = 1 + \frac{1}{2}\omega^{\alpha\beta}(x)\sigma_{\alpha\beta} \tag{4.40}$$

with $\sigma_{\alpha\beta}$ being constant, anti-symmetric matrices, satisfying equation (4.14).

Now, $V^\alpha_{\ \mu}(x)$ transforms as,

$$V^\alpha_{\ \mu}(x) \to \Lambda^\alpha{}_\beta(x) V^\beta_{\ \mu}(x)$$

But $\Lambda^\alpha{}_\beta(x) = \delta^\alpha{}_\beta + \omega^\alpha{}_\beta(x)$, and the above transformation becomes,

$$\begin{aligned}
V^\alpha_{\ \mu}(x) &\to \big(\delta^\alpha{}_\beta + \omega^\alpha{}_\beta(x)\big)V^\beta_{\ \mu}(x) \\
\implies V^\alpha_{\ \mu}(x) &\to V^\alpha_{\ \mu}(x) + \omega^\alpha{}_\beta(x)V^\beta_{\ \mu}(x)
\end{aligned} \tag{4.41}$$

We will now consider the Lorentz transformation of $V_\beta{}^\nu \frac{\partial}{\partial x^\mu}\big(V_{\alpha\nu}\big)$. The transformations of $V_\beta{}^\nu(x)$ and $V_{\alpha\nu}(x)$ are as follows,

$$V_\beta{}^\nu(x) \to V_\beta{}^\nu + \omega_\beta{}^\gamma V_\gamma{}^\nu \tag{4.42}$$

$$V_{\alpha\nu}(x) \to V_{\alpha\nu} + \omega_\alpha{}^\gamma V_{\gamma\nu} \tag{4.43}$$

Thus, $V_\beta{}^\nu(x)\partial_\mu\big(V_{\alpha\nu}(x)\big)$ transforms as follows,

$$\begin{aligned}
V_\beta{}^\nu(x)\partial_\mu\big(V_{\alpha\nu}(x)\big) \rightarrow & \Big[V_\beta{}^\nu + \omega_\beta{}^\gamma V_\gamma{}^\nu\Big]\frac{\partial}{\partial x^\mu}\Big[V_{\alpha\nu} + \omega_\alpha{}^\gamma V_{\gamma\nu}\Big] \\
\equiv & \Big[V_\beta{}^\nu \frac{\partial}{\partial x^\mu}\Big(V_{\alpha\nu}\Big) + V_\beta{}^\nu \frac{\partial}{\partial x^\mu}\Big(\omega_\alpha{}^\gamma V_{\gamma\nu}\Big) + \omega_\beta{}^\gamma V_\gamma{}^\nu \frac{\partial}{\partial x^\mu}\Big(V_{\alpha\nu}\Big) \\
& + \omega_\beta{}^\gamma V_\gamma{}^\nu \frac{\partial}{\partial x^\mu}\Big(\omega_\alpha{}^\gamma V_{\gamma\nu}\Big)\Big] \\
\equiv & V_\beta{}^\nu \frac{\partial}{\partial x^\mu}\Big(V_{\alpha\nu}\Big) + V_\beta{}^\nu \omega_\alpha{}^\gamma \frac{\partial}{\partial x^\mu}\Big(V_{\gamma\nu}\Big) + V_\beta{}^\nu V_{\gamma\nu}\frac{\partial}{\partial x^\mu}\Big(\omega_\alpha{}^\gamma\Big) \\
& + \omega_\beta{}^\gamma V_\gamma{}^\nu \frac{\partial}{\partial x^\mu}\Big(V_{\alpha\nu}\Big) \\
\equiv & V_\beta{}^\nu \frac{\partial}{\partial x^\mu}\Big(V_{\alpha\nu}\Big) + \omega_\alpha{}^\gamma V_\beta{}^\nu \frac{\partial}{\partial x^\mu}\Big(V_{\gamma\nu}\Big) + \omega_\beta{}^\gamma V_\gamma{}^\nu \frac{\partial}{\partial x^\mu}\Big(V_{\alpha\nu}\Big) + \frac{\partial}{\partial x^\mu}\Big(\omega_{\alpha\beta}\Big)
\end{aligned} \tag{4.44}$$

Using equation (4.38), the transformation of Γ_μ under an infinitesimal Lorentz transformation is as follows,

$$\begin{aligned}
\Gamma_\mu \rightarrow & D(1+\omega(x))\Gamma_\mu(x)D^{-1}(1+\omega(x)) - \frac{\partial}{\partial x^\mu}\Big[D(1+\omega(x))\Big]\Big(D^{-1}(1+\omega(x))\Big) \\
\equiv & \Big[1+\frac{1}{2}\omega^{\alpha\beta}\sigma_{\alpha\beta}\Big]\Gamma_\mu\Big[1-\frac{1}{2}\omega^{\alpha\beta}\sigma_{\alpha\beta}\Big] - \frac{\partial}{\partial x^\mu}\Big[1+\frac{1}{2}\omega^{\alpha\beta}\sigma_{\alpha\beta}\Big]\Big(1-\frac{1}{2}\omega^{\alpha\beta}\sigma_{\alpha\beta}\Big) \\
\equiv & \Gamma_\mu + \frac{1}{2}\omega^{\alpha\beta}\sigma_{\alpha\beta}\Gamma_\mu - \frac{1}{2}\omega^{\alpha\beta}\Gamma_\mu\sigma_{\alpha\beta} - \frac{1}{2}\Big[\frac{\partial\omega^{\alpha\beta}}{\partial x^\mu}\Big]\sigma_{\alpha\beta} \\
\equiv & \Gamma_\mu + \frac{1}{2}\omega^{\alpha\beta}(x)[\sigma_{\alpha\beta},\Gamma_\mu] - \frac{\sigma_{\alpha\beta}}{2}\frac{\partial}{\partial x^\mu}\Big(\omega^{\alpha\beta}(x)\Big)
\end{aligned} \tag{4.45}$$

The matrix Γ_μ that satisfies the above transformation property, will have the following form,

$$\Gamma_\mu(x) = \frac{1}{2}\omega^{\alpha\beta}V_\alpha{}^\nu(x)V_{\beta\nu;\mu} \tag{4.46}$$

Thus, in order to take into account the effect of gravitational field on a physical system, we replace the $\frac{\partial}{\partial x^\alpha}$ in its action or field equation with the covariant derivative, given by,

$$\mathscr{D}_\alpha \equiv V_\alpha{}^\mu\frac{\partial}{\partial x^\mu} + \frac{1}{2}\sigma^{\beta\gamma}V_\alpha{}^\mu V_\beta{}^\nu V_{\gamma\nu;\mu} \tag{4.47}$$

Here, the semi-colon denotes the usual covariant vector derivative involving Christoffel symbols,

$$V_{\gamma\nu;\mu} = \frac{\partial V_{\gamma\nu}}{\partial x^\mu} - \Gamma^\alpha{}_{\gamma\mu}V_{\alpha\nu} - \Gamma^\beta{}_{\nu\mu}V_{\gamma\beta} \tag{4.48}$$

Now, the Dirac equation in flat spacetime was written as,

$$i\gamma_{(0)}{}^\mu\partial_\mu\psi(x) = m\psi(x)$$

The covariant form of Dirac equation is then given by,

$$i\gamma_{(0)}{}^\mu\mathscr{D}_\mu{}^*\psi = m^*\psi \tag{4.49}$$

If we define the Gamma matrices to be,

$$\gamma^{\mu}(x) = {\gamma_{(0)}}^{\beta} {V_{\beta}}^{\mu} \tag{4.50}$$

and with the wave function of the electron being the same in all global coordinate frames and therefore, ${}^*\psi = \psi$, then the covariant Dirac equation thus, becomes,

$$i\gamma^{\mu}\mathscr{D}_{\mu}\psi(x) = m\psi(x) \tag{4.51}$$

4.3 Spin Connection

We will now, show that the spin connection Γ_{μ}, in the covariant derivative $\mathscr{D}_{\mu}$, defined below,

$$\mathscr{D}_{\mu} \equiv \partial_{\mu} + \Gamma_{\mu}$$

does not contribute to the acceleration of an electron in a uniform gravitational field. The spin connection will be evaluated using the Schwarzschild metric, we have obtained in the second chapter.

For this, we will construct a local inertial reference frame ${\xi_X}^{\alpha}$, at every point X, in spacetime such that its corresponding metric $g^{\mu\nu}$ is the Minkowski metric with its derivatives vanishing at the origin. The vierbein can be determined by describing the coordinates of the local inertial frame ${\xi_X}^{\alpha}(x)$, that is centered at the point X, in terms of the global coordinates x^{μ}, using the following set of relations,

$$\begin{aligned}
{\xi_X}^0 &= (1+\Phi_0)\tilde{x}^0 + \nabla\Phi_G \tilde{x}^0\tilde{x}^3 \\
{\xi_X}^1 &= (1-\Phi_0)\tilde{x}^1 - \nabla\Phi_G \tilde{x}^1\tilde{x}^3 \\
{\xi_X}^2 &= (1-\Phi_0)\tilde{x}^2 - \nabla\Phi_G \tilde{x}^2\tilde{x}^3 \\
{\xi_X}^3 &= (1-\Phi_0)\tilde{x}^3 + \frac{1}{2}\nabla\Phi_G\left(\tilde{x}^0\tilde{x}^0 + \tilde{x}^1\tilde{x}^1 + \tilde{x}^2\tilde{x}^2 - \tilde{x}^3\tilde{x}^3\right)
\end{aligned}$$

where, $\tilde{x}^{\mu} = x^{\mu} - X^{\mu}$, the distance of any point in spacetime from the point X with x^{μ} being the global coordinates of the former point and X^{μ} being the global coordinates of the point X, and $\Phi_0 = \Phi_G(X)$. Also, $\nabla\Phi_G$ is a constant, as shown in the previous chapter.

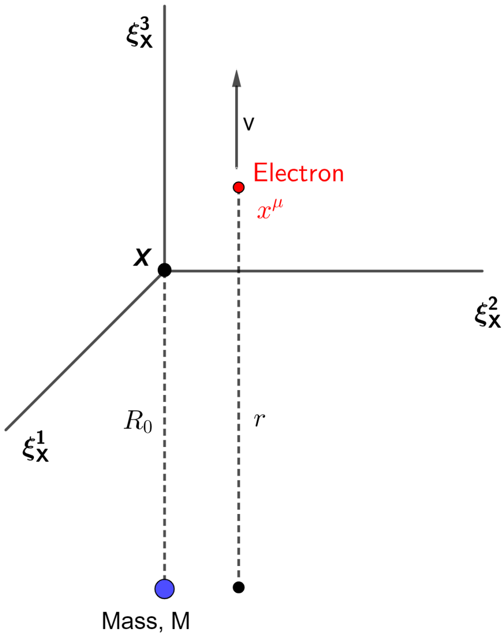

Figure 4.1: This figure shows the local inertial frame ${\xi_X}^\alpha$ at point X, along with the electron situated at the global spacetime coordinates x^μ. Note that the time axis of the local inertial frame ${\xi_X}^0(x)$ has not been shown.

From the tetrad formalism, the vierbein is defined as,

$$V^{\alpha}_{\ \mu}(X) = \left(\frac{\partial {\xi_X}^\alpha(x)}{\partial x^\mu}\right)\bigg|_{x=X}$$

Therefore, we can evaluate each vierbein as follows,

$$\begin{aligned}
V^0_{\ 0} &= \left(\frac{\partial \xi_X{}^0(x)}{\partial x^0}\right)\bigg|_{x=X} \\
&= \left[(1+\Phi_0) + \nabla\Phi_G \tilde{x}^3\right]\bigg|_{x=X} \\
&= \left[1+\Phi_0 + \nabla\Phi_G\big(X^3 - X^3\big)\right] \\
\therefore V^0_{\ 0} &= 1+\Phi_0
\end{aligned}$$

$$\begin{aligned}
V^0_{\ 1} &= \left(\frac{\partial \xi_X{}^0(x)}{\partial x^1}\right)\bigg|_{x=X} \\
&= \frac{\partial}{\partial x^1}\left[(1+\Phi_0)\tilde{x}^0 + \nabla\Phi_G \tilde{x}^0\tilde{x}^3\right]\bigg|_{x=X} \\
\therefore V^0_{\ 1} &= 0
\end{aligned}$$

$$\begin{aligned}
V^0_{\ 2} &= \left(\frac{\partial \xi_X{}^0(x)}{\partial x^2}\right)\bigg|_{x=X} \\
&= \frac{\partial}{\partial x^2}\left[(1+\Phi_0)\tilde{x}^0 + \nabla\Phi_G \tilde{x}^0\tilde{x}^3\right]\bigg|_{x=X} \\
\therefore V^0_{\ 2} &= 0
\end{aligned}$$

$$\begin{aligned}
V^0_{\ 3} &= \left(\frac{\partial \xi_X{}^0(x)}{\partial x^3}\right)\bigg|_{x=X} \\
&= \frac{\partial}{\partial x^3}\left[(1+\Phi_0)\tilde{x}^0 + \nabla\Phi_G \tilde{x}^0\tilde{x}^3\right]\bigg|_{x=X} \\
&= \left[\nabla\Phi_G \tilde{x}^0\right]\bigg|_{x=X} \\
\therefore V^0_{\ 3} &= 0
\end{aligned}$$

$$\begin{aligned}
V^1_{\ 0} &= \left(\frac{\partial \xi_X{}^1(x)}{\partial x^0}\right)\bigg|_{x=X} \\
&= \frac{\partial}{\partial x^0}\left[(1-\Phi_0)\tilde{x}^1 - \nabla\Phi_G \tilde{x}^1\tilde{x}^3\right]\bigg|_{x=X} \\
\therefore V^1_{\ 0} &= 0
\end{aligned}$$

$$
\begin{aligned}
V^{1}{}_{1} &= \left(\frac{\partial {\xi_X}^{1}(x)}{\partial x^{1}}\right)\bigg|_{x=X} \\
&= \frac{\partial}{\partial x^{1}}\left[(1-\Phi_0)\tilde{x}^{1} - \nabla\Phi_G \tilde{x}^{1}\tilde{x}^{3}\right]\bigg|_{x=X} \\
&= \left[(1-\Phi_0) - \nabla\Phi_G\big(\tilde{x}^{3}\big)\right]\bigg|_{x=X} \\
\therefore V^{1}{}_{1} &= (1-\Phi_0)
\end{aligned}
$$

$$
\begin{aligned}
V^{1}{}_{2} &= \left(\frac{\partial {\xi_X}^{1}(x)}{\partial x^{2}}\right)\bigg|_{x=X} \\
&= \frac{\partial}{\partial x^{2}}\left[(1-\Phi_0)\tilde{x}^{1} - \nabla\Phi_G \tilde{x}^{1}\tilde{x}^{3}\right]\bigg|_{x=X} \\
\therefore V^{1}{}_{2} &= 0
\end{aligned}
$$

$$
\begin{aligned}
V^{1}{}_{3} &= \left(\frac{\partial {\xi_X}^{1}(x)}{\partial x^{3}}\right)\bigg|_{x=X} \\
&= \frac{\partial}{\partial x^{3}}\left[(1-\Phi_0)\tilde{x}^{1} - \nabla\Phi_G \tilde{x}^{1}\tilde{x}^{3}\right]\bigg|_{x=X} \\
\therefore V^{1}{}_{3} &= 0
\end{aligned}
$$

$$
\begin{aligned}
V^{2}{}_{0} &= \left(\frac{\partial {\xi_X}^{2}(x)}{\partial x^{0}}\right)\bigg|_{x=X} \\
&= \frac{\partial}{\partial x^{0}}\left[(1-\Phi_0)\tilde{x}^{2} - \nabla\Phi_G \tilde{x}^{2}\tilde{x}^{3}\right]\bigg|_{x=X} \\
\therefore V^{2}{}_{0} &= 0
\end{aligned}
$$

$$
\begin{aligned}
V^{2}{}_{1} &= \left(\frac{\partial {\xi_X}^{2}(x)}{\partial x^{1}}\right)\bigg|_{x=X} \\
&= \frac{\partial}{\partial x^{1}}\left[(1-\Phi_0)\tilde{x}^{2} - \nabla\Phi_G \tilde{x}^{2}\tilde{x}^{3}\right]\bigg|_{x=X} \\
\therefore V^{2}{}_{1} &= 0
\end{aligned}
$$

$$
\begin{aligned}
V^{2}{}_{2} &= \left(\frac{\partial {\xi_X}^{2}(x)}{\partial x^{2}}\right)\bigg|_{x=X} \\
&= \frac{\partial}{\partial x^{2}}\left[(1-\Phi_0)\tilde{x}^{2} - \nabla\Phi_G \tilde{x}^{2}\tilde{x}^{3}\right]\bigg|_{x=X} \\
&= \left[(1-\Phi_0) - \nabla\Phi_G \tilde{x}^{3}\right]\bigg|_{x=X} \\
\therefore V^{2}{}_{2} &= (1-\Phi_0)
\end{aligned}
$$

$$
\begin{aligned}
V^{2}{}_{3} &= \left(\frac{\partial {\xi_X}^{2}(x)}{\partial x^{3}}\right)\bigg|_{x=X} \\
&= \frac{\partial}{\partial x^{3}}\left[(1-\Phi_0)\tilde{x}^{2} - \nabla\Phi_G \tilde{x}^{2}\tilde{x}^{3}\right]\bigg|_{x=X} \\
&= \left[-\nabla\Phi_G \tilde{x}^{2}\right]\bigg|_{x=X} \\
\therefore V^{2}{}_{3} &= 0
\end{aligned}
$$

$$
\begin{aligned}
V^3_{\ 0} &= \left(\frac{\partial {\xi_X}^3(x)}{\partial x^0}\right)\bigg|_{x=X} \\
&= \frac{\partial}{\partial x^0}\ \left[(1-\Phi_0)\tilde{x}^3 + \frac{1}{2}\nabla\Phi_G\big(\tilde{x}^0\tilde{x}^0 + \tilde{x}^1\tilde{x}^1 + \tilde{x}^2\tilde{x}^2 - \tilde{x}^3\tilde{x}^3\big)\right]\bigg|_{x=X} \\
&= \frac{1}{2}\nabla\Phi_G\Big[2\tilde{x}^0\Big]\bigg|_{x=X} \\
&= \nabla\Phi_G\Big[x^0 - X^0\Big]\bigg|_{x=X} \\
\therefore V^3_{\ 0} &= 0
\end{aligned}
$$

$$
\begin{aligned}
V^3_{\ 1} &= \left(\frac{\partial {\xi_X}^3(x)}{\partial x^1}\right)\bigg|_{x=X} \\
&= \frac{\partial}{\partial x^1}\ \left[(1-\Phi_0)\tilde{x}^3 + \frac{1}{2}\nabla\Phi_G\big(\tilde{x}^0\tilde{x}^0 + \tilde{x}^1\tilde{x}^1 + \tilde{x}^2\tilde{x}^2 - \tilde{x}^3\tilde{x}^3\big)\right]\bigg|_{x=X} \\
&= \frac{\nabla\Phi_G}{2}\Big[2\tilde{x}^1\Big]\bigg|_{x=X} \\
&= \nabla\Phi_G\Big[x^1 - X^1\Big]\bigg|_{x=X} \\
\therefore V^3_{\ 1} &= 0
\end{aligned}
$$

$$
\begin{aligned}
V^3_{\ 2} &= \left(\frac{\partial {\xi_X}^3(x)}{\partial x^2}\right)\bigg|_{x=X} \\
&= \frac{\partial}{\partial x^2}\ \left[(1-\Phi_0)\tilde{x}^3 + \frac{1}{2}\nabla\Phi_G\big(\tilde{x}^0\tilde{x}^0 + \tilde{x}^1\tilde{x}^1 + \tilde{x}^2\tilde{x}^2 - \tilde{x}^3\tilde{x}^3\big)\right]\bigg|_{x=X} \\
&= \frac{\nabla\Phi_G}{2}\Big[2\tilde{x}^2\Big]\bigg|_{x=X} \\
&= \nabla\Phi_G\Big[x^2 - X^2\Big]\bigg|_{x=X} \\
\therefore V^3_{\ 2} &= 0
\end{aligned}
$$

$$
\begin{aligned}
V^3_{\ 3} &= \left(\frac{\partial {\xi_X}^3(x)}{\partial x^3}\right)\bigg|_{x=X} \\
&= \frac{\partial}{\partial x^3}\ \left[(1-\Phi_0)\tilde{x}^3 + \frac{1}{2}\nabla\Phi_G\big(\tilde{x}^0\tilde{x}^0 + \tilde{x}^1\tilde{x}^1 + \tilde{x}^2\tilde{x}^2 - \tilde{x}^3\tilde{x}^3\big)\right]\bigg|_{x=X} \\
&= \left[\big(1-\Phi_0\big) + \frac{\nabla\Phi_G}{2}\left(2\big(x^3 - X^3\big)\right)\right]\bigg|_{x=X} \\
\therefore V^3_{\ 3} &= (1-\Phi_0)
\end{aligned}
$$

The Schwarzschild metric with the indices raised and lowered are as follows,

$$g^{\mu\nu} = \begin{pmatrix} (1-2\Phi_G) & 0 & 0 & 0 \\ 0 & -(1+2\Phi_G) & 0 & 0 \\ 0 & 0 & -(1+2\Phi_G) & 0 \\ 0 & 0 & 0 & -(1+2\Phi_G) \end{pmatrix}$$

$$g_{\mu\nu} = \begin{pmatrix} \frac{1}{(1-2\Phi_G)} & 0 & 0 & 0 \\ 0 & -\frac{1}{(1+2\Phi_G)} & 0 & 0 \\ 0 & 0 & -\frac{1}{(1+2\Phi_G)} & 0 \\ 0 & 0 & 0 & -\frac{1}{(1+2\Phi_G)} \end{pmatrix}$$

The corresponding Christoffel symbols to this metric is calculated using the formula,

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{g^{\mu\lambda}}{2}\left[\frac{\partial g_{\alpha\lambda}}{\partial x^{\beta}} + \frac{\partial g_{\beta\lambda}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\lambda}}\right] \tag{4.52}$$

Now, $\Gamma^{0}{}_{00}$ is evaluated as,

$$\begin{aligned} \Gamma^{0}{}_{00} &= \frac{g^{0\lambda}}{2}\left[\frac{\partial g_{0\lambda}}{\partial x^{0}} + \frac{\partial g_{0\lambda}}{\partial x^{0}} - \frac{\partial g_{00}}{\partial x^{\lambda}}\right] \\ &= \frac{g^{00}}{2}\left[-\frac{\partial g_{00}}{\partial x^{0}}\right] \\ \therefore \Gamma^{0}{}_{00} &= 0 \end{aligned}$$

where, $\frac{\partial g_{0\lambda}}{\partial x^0} = 0$. Now, for $\Gamma^{0}{}_{0i}$,

$$\begin{aligned} \Gamma^{0}{}_{0i} &= \frac{g^{0\lambda}}{2}\left[\frac{\partial g_{0\lambda}}{\partial x^{i}} + \frac{\partial g_{i\lambda}}{\partial x^{0}} - \frac{\partial g_{0i}}{\partial x^{\lambda}}\right] \\ &= \frac{g^{00}}{2}\left[\frac{\partial g_{00}}{\partial x^{i}}\right] \end{aligned}$$

Now, $\frac{\partial g_{00}}{\partial x^1} = \frac{\partial g_{00}}{\partial x^2} = 0$

$$\therefore \Gamma^{0}{}_{01} = \Gamma^{0}{}_{02} = 0$$

And for the case of $\Gamma^{0}{}_{03}$,

$$\begin{aligned} \Gamma^{0}{}_{03} = \Gamma^{0}{}_{30} &= \frac{(1-2\Phi_G)}{2}\frac{\partial}{\partial x^3}\left(\frac{1}{(1-2\Phi_G)}\right) \\ &= \frac{(1-2\Phi_G)}{2}\frac{1}{(1-2\Phi_G)^2}\left(2\frac{\partial\Phi_G}{\partial x^3}\right) \\ &= \frac{\partial\Phi_G}{\partial x^i}(1-2\Phi_G)^{-1} \\ &= \nabla\Phi_G(1+2\Phi_G+\dots) \\ \therefore \Gamma^{0}{}_{03} = \Gamma^{0}{}_{30} &\approx \nabla\Phi_G \text{ (Ignoring higher order terms)} \end{aligned}$$

For the case of $\Gamma^{0}{}_{ij}$, we have,

$$\begin{aligned} \Gamma^{0}{}_{ij} &= \frac{g^{0\lambda}}{2}\left[\frac{\partial g_{i\lambda}}{\partial x^{j}} + \frac{\partial g_{j\lambda}}{\partial x^{i}} - \frac{\partial g_{ij}}{\partial x^{\lambda}}\right] \\ &= \frac{g^{00}}{2}\left[\frac{\partial g_{i0}}{\partial x^{j}} + \frac{\partial g_{j0}}{\partial x^{i}} - \frac{\partial g_{ij}}{\partial x^{0}}\right] \end{aligned}$$

But,

$$\frac{\partial g_{i0}}{\partial x^j} = \frac{\partial g_{j0}}{\partial x^i} = \frac{\partial g_{ij}}{\partial x^0} = 0$$

$$\therefore \Gamma^0{}_{ij} = 0$$

Now, for the case of $\Gamma^1{}_{00}$,

$$\begin{aligned}
\Gamma^1{}_{00} &= \frac{g^{1\lambda}}{2}\left[\frac{\partial g_{0\lambda}}{\partial x^0} + \frac{\partial g_{0\lambda}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\lambda}\right] \\
&= \frac{g^{11}}{2}\left[\frac{\partial g_{01}}{\partial x^0} + \frac{\partial g_{01}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^1}\right]
\end{aligned}$$

But,

$$\frac{\partial g_{01}}{\partial x^0} = \frac{\partial g_{00}}{\partial x^1} = 0$$

$$\therefore \Gamma^1{}_{00} = 0$$

For the case of $\Gamma^1{}_{i0}$,

$$\begin{aligned}
\Gamma^1{}_{i0} &= \frac{g^{1\lambda}}{2}\left[\frac{\partial g_{i\lambda}}{\partial x^0} + \frac{\partial g_{0\lambda}}{\partial x^i} - \frac{\partial g_{i0}}{\partial x^\lambda}\right] \\
&= \frac{g^{11}}{2}\left[\frac{\partial g_{i1}}{\partial x^0} + \frac{\partial g_{01}}{\partial x^i} - \frac{\partial g_{i0}}{\partial x^1}\right]
\end{aligned}$$

In this case too,

$$\frac{\partial g_{i1}}{\partial x^0} = \frac{\partial g_{01}}{\partial x^i} = \frac{\partial g_{i0}}{\partial x^1} = 0$$

$$\therefore \Gamma^1{}_{i0} = \Gamma^1{}_{0i} = 0$$

In case of $\Gamma^1{}_{ij}$,

$$\begin{aligned}
\Gamma^1{}_{ij} =& \frac{g^{1\lambda}}{2}\left[\frac{\partial g_{i\lambda}}{\partial x^j} + \frac{\partial g_{j\lambda}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\lambda}\right] \\
=& \frac{g^{11}}{2}\left[\frac{\partial g_{i1}}{\partial x^j} + \frac{\partial g_{j1}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^1}\right]
\end{aligned}$$

Here, $\frac{\partial g_{ij}}{\partial x^1} = 0$. If $i = 1$ and $j = 3$, (or vice-versa),

$$\begin{aligned}
\Gamma^1{}_{13} = \Gamma^1{}_{31} &= \frac{g^{11}}{2}\left[\frac{\partial g_{11}}{\partial x^3}\right] \\
&= -\frac{(1+2\Phi_G)}{2}\frac{2}{(1+2\Phi_G)^2}\frac{\partial \Phi_G}{\partial x^3} \\
&= -[1 - 2\Phi_G + \dots]\nabla\Phi_G \\
\therefore \Gamma^1{}_{13} = \Gamma^1{}_{31} &\approx -\nabla\Phi_G
\end{aligned}$$

For all the other values of i and j,

$$\frac{\partial g_{i1}}{\partial x^j} = \frac{\partial g_{j1}}{\partial x^i} = 0$$

When $i \neq 1$ and $j \neq 3$ or vice versa, we have

$$\therefore \Gamma^1{}_{ij} = 0$$

Now, for the case of $\Gamma^2{}_{00}$,

$$\Gamma^2{}_{00} = \frac{g^{2\lambda}}{2}\left[\frac{\partial g_{0\lambda}}{\partial x^0} + \frac{\partial g_{0\lambda}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\lambda}\right]$$

However, for any value of λ,

$$\frac{\partial g_{0\lambda}}{\partial x^0} = \frac{\partial g_{00}}{\partial x^\lambda} = 0$$

$$\therefore \Gamma^2{}_{00} = 0$$

For the case of $\Gamma^2{}_{0i}$, we have,

$$\begin{aligned}\Gamma^2{}_{0i} &= \frac{g^{2\lambda}}{2}\left[\frac{\partial g_{0\lambda}}{\partial x^i} + \frac{\partial g_{i\lambda}}{\partial x^0} - \frac{\partial g_{0i}}{\partial x^\lambda}\right] \\ &= \frac{g^{22}}{2}\left[\frac{\partial g_{02}}{\partial x^i} + \frac{\partial g_{i2}}{\partial x^0} - \frac{\partial g_{0i}}{\partial x^2}\right]\end{aligned}$$

However, in this case also,

$$\frac{\partial g_{02}}{\partial x^i} = \frac{\partial g_{i2}}{\partial x^0} = \frac{\partial g_{0i}}{\partial x^2} = 0$$

$$\therefore \Gamma^2{}_{0i} = 0$$

Now, for $\Gamma^2{}_{ij}$, we have,

$$\begin{aligned}\Gamma^2{}_{ij} &= \frac{g^{2\lambda}}{2}\left[\frac{\partial g_{i\lambda}}{\partial x^j} + \frac{\partial g_{j\lambda}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\lambda}\right] \\ &= \frac{g^{22}}{2}\left[\frac{\partial g_{i2}}{\partial x^j} + \frac{\partial g_{j2}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^2}\right]\end{aligned}$$

Now, here $\frac{\partial g_{ij}}{\partial x^2} = 0$. If $i = 2$ and $j = 3$, or vice versa, then we have,

$$\begin{aligned}\Gamma^2{}_{23} = \Gamma^2{}_{32} &= \frac{g^{22}}{2}\left[\frac{\partial g_{22}}{\partial x^3} + \frac{\partial g_{32}}{\partial x^2}\right] \\ &= \frac{(1+2\Phi_G)}{2}\left[\frac{\partial}{\partial x^3}\left(\frac{1}{(1+2\Phi_G)}\right)\right] \\ &= -\frac{(1+2\Phi_G)}{2}\frac{1}{(1+2\Phi_G)^2}(2\nabla\Phi_G) \\ &= -\nabla\Phi_G(1+2\Phi_G)^{-1} \\ &= -\nabla\Phi_G\Big(1 - 2\Phi_G + (2\Phi_G)^2 - \dots\Big)\end{aligned}$$

$$\therefore \Gamma^2{}_{23} \approx \Gamma^2{}_{32} = -\nabla\Phi_G$$

For all the other values of i and j, we have,

$$\frac{\partial g_{i2}}{\partial x^j} = \frac{\partial g_{j2}}{\partial x^i} = 0$$

When, $i \neq 2$ and $j \neq 3$ or vice versa,

$$\therefore \Gamma^2{}_{ij} = 0$$

Now, let us consider $\Gamma^3{}_{00}$,

$$\begin{aligned} \Gamma^3{}_{00} &= \frac{g^{3\lambda}}{2}\left[\frac{\partial g_{0\lambda}}{\partial x^0} + \frac{\partial g_{0\lambda}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\lambda}\right] \\ &= \frac{g^{33}}{2}\left[\frac{\partial g_{03}}{\partial x^0} + \frac{\partial g_{03}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^3}\right] \end{aligned}$$

But,

$$\frac{\partial g_{03}}{\partial x^0} = 0$$

So the value of $\Gamma^3{}_{00}$ becomes,

$$\begin{aligned} \Gamma^3{}_{00} &= \frac{(1+2\Phi_G)}{2}\frac{\partial}{\partial x^3}\left(\frac{1}{1-2\Phi_G}\right) \\ &= \frac{(1+2\Phi_G)}{2}\frac{1}{(1-2\Phi_G)^2}\left(2\frac{\partial \Phi_G}{\partial x^3}\right) \\ &= (1+2\Phi_G)(1-2\Phi_G)^{-2}\nabla\Phi_G \\ &= (1+2\Phi_G)\Big(1+2\Phi_G+3(\Phi_G)^2+\dots\Big)\nabla\Phi_G \\ &= \Big(1+2\Phi_G+3(\Phi_G)^2+\dots+2\Phi_G+4\Phi_G^2+6\Phi_G^3+\dots\Big)\nabla\Phi_G \\ \therefore \Gamma^3{}_{00} &\approx \nabla\Phi_G \end{aligned}$$

For $\Gamma^3{}_{0i}$, we have,

$$\begin{aligned} \Gamma^3{}_{0i} &= \frac{g^{3\lambda}}{2}\left[\frac{\partial g_{0\lambda}}{\partial x^i} + \frac{\partial g_{\lambda i}}{\partial x^0} - \frac{\partial g_{0i}}{\partial x^\lambda}\right] \\ &= \frac{g^{33}}{2}\left[\frac{\partial g_{03}}{\partial x^i} + \frac{\partial g_{3i}}{\partial x^0} - \frac{\partial g_{0i}}{\partial x^3}\right] \end{aligned}$$

Here also, since i only takes values of $1, 2, 3$, we have,

$$\frac{\partial g_{03}}{\partial x^i} = \frac{\partial g_{3i}}{\partial x^0} = \frac{\partial g_{0i}}{\partial x^3} = 0$$

$$\therefore \Gamma^3{}_{0i} = \Gamma^3{}_{i0} = 0$$

For $\Gamma^3{}_{ij}$, we have,

$$\begin{aligned} \Gamma^3{}_{ij} &= \frac{g^{3\lambda}}{2}\left[\frac{\partial g_{i\lambda}}{\partial x^j} + \frac{\partial g_{\lambda j}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\lambda}\right] \\ &= \frac{g^{33}}{2}\left[\frac{\partial g_{i3}}{\partial x^j} + \frac{\partial g_{3j}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^3}\right] \end{aligned}$$

Now, if we have $i = j = 3$, then,

$$\begin{aligned}
\Gamma^3{}_{33} &= \frac{g^{33}}{2}\left[\frac{\partial g_{33}}{\partial x^3} + \frac{\partial g_{33}}{\partial x^3} - \frac{\partial g_{33}}{\partial x^3}\right] \\
&= \frac{g^{33}}{2}\left[\frac{\partial g_{33}}{\partial x^3}\right] \\
&= \frac{(1+2\Phi_G)}{2}\frac{\partial}{\partial x^3}\left(\frac{1}{1+2\Phi_G}\right) \\
&= \frac{(1+2\Phi_G)}{(1+2\Phi_G)^2}\left(-\frac{\partial \Phi_G}{\partial x^3}\right) \\
&= (1+2\Phi_G)^{-1}[-\nabla\Phi_G] \\
&= \left(1-2\Phi_G+4\Phi_G^2\right)[-\nabla\Phi_G] \\
\therefore \Gamma^3{}_{33} &\approx -\nabla\Phi_G
\end{aligned}$$

Also, if $i = j = 1$, then,

$$\frac{\partial g_{3i}}{\partial x^j} = \frac{\partial g_{3j}}{\partial x^i} = 0$$

$$\begin{aligned}
\therefore \Gamma^3{}_{11} &= \frac{g^{33}}{2}\left[-\frac{\partial g_{11}}{\partial x^3}\right] \\
&= \frac{(1+2\Phi_G)}{2}\left[\frac{\partial}{\partial x^3}\left(\frac{1}{1+2\Phi_G}\right)\right] \\
&= \frac{(1+2\Phi_G)}{(1+2\Phi_G)^2}\left[\frac{\partial \Phi_G}{\partial x^3}\right] \\
&= (1+2\Phi_G)^{-1}\nabla\Phi_G \\
&= \left(1-2\Phi_G+4\Phi_G^2+\ldots\right)\nabla\Phi_G \\
\therefore \Gamma^3{}_{11} &\approx \nabla\Phi_G
\end{aligned}$$

Finally, if $i = j = 2$, then,

$$\frac{\partial g_{3i}}{\partial x^j} = \frac{\partial g_{3j}}{\partial x^i} = 0$$

$$\begin{aligned}
\therefore \Gamma^3{}_{22} &= \frac{g^{33}}{2}\left[-\frac{\partial g_{22}}{\partial x^3}\right] \\
&= \frac{(1+2\Phi_G)}{2}\left[\frac{\partial}{\partial x^3}\left(\frac{-1}{1+2\Phi_G}\right)\right] \\
&= \frac{(1+2\Phi_G)}{(1+2\Phi_G)^2}\left[\frac{\partial \Phi_G}{\partial x^3}\right] \\
&= (1+2\Phi_G)^{-1}\nabla\Phi_G \\
&= \left(1-2\Phi_G+4\Phi_G^2+\ldots\right)\nabla\Phi_G \\
\therefore \Gamma^3{}_{22} &\approx \nabla\Phi_G
\end{aligned}$$

Now, the spin connection is defined as,

$$\Gamma_\mu = \frac{1}{2}\sigma^{\alpha\beta}V_\alpha{}^\nu V_{\beta\nu;\mu} \tag{4.53}$$

where, $\sigma^{\alpha\beta}$ is defined as,

$$\sigma^{\alpha\beta} = \frac{1}{4}\left[\gamma_{(0)}{}^{\alpha}, \gamma_{(0)}{}^{\beta}\right] = \frac{1}{4}\left[\gamma_{(0)}{}^{\alpha}\gamma_{(0)}{}^{\beta} - \gamma_{(0)}{}^{\beta}\gamma_{(0)}{}^{\alpha}\right] \tag{4.54}$$

and the semi-colon represents the usual covariant derivative, defined as follows,

$$V_{\beta\nu;\mu} = \partial_\mu V_{\beta\nu} - \Gamma^{\alpha}{}_{\mu\nu} V_{\alpha\beta} - \Gamma^{\alpha}{}_{\mu\beta} V_{\alpha\nu}$$

The general rule for the lowering and raising of the tetrad indices are as follows,

$$V_{\beta}{}^{\nu} = \eta_{\alpha\beta} g^{\mu\nu} V^{\alpha}_{\ \ \mu}$$

where $\eta_{\alpha\beta}$ is the Minkowski metric and the metric $g^{\mu\nu}$ in this case is the Schwarzschild metric. For the case of only lowering the indices, we use the formula,

$$V_{\beta\mu} = \eta_{\alpha\beta} V^{\alpha}_{\ \ \mu}$$

Now, V_{00} and $V_0{}^0$ will evaluate to,

$$\begin{aligned} V_{00} &= \eta_{\alpha 0} V^{\alpha}_{\ \ 0} = \eta_{00} V^{0}_{\ \ 0} = (1 + \Phi_0) \\ V_0{}^0 &= \eta_{\alpha 0} g^{\mu 0} V^{\alpha}_{\ \ \mu} = \eta_{00} g^{00} V^{0}_{\ \ 0} = (1 - 2\Phi_G)(1 + \Phi_0) \end{aligned}$$

We will now evaluate the rest of the vierbeins components for the same set of lowered/raised indices, in a similar manner,

$$\begin{aligned} V_{11} &= \eta_{\alpha 1} V^{\alpha}_{\ \ 1} = \eta_{11} V^{1}_{\ \ 1} = -(1 - \Phi_0) \\ V_1{}^1 &= \eta_{\alpha 1} g^{\mu 1} V^{\alpha}_{\ \ \mu} = \eta_{11} g^{11} V^{1}_{\ \ 1} = (1 + 2\Phi_G)(1 - \Phi_0) \\ V_{22} &= \eta_{\alpha 2} V^{\alpha}_{\ \ 2} = \eta_{22} V^{2}_{\ \ 2} = -(1 - \Phi_0) \\ V_2{}^2 &= \eta_{\alpha 2} g^{\mu 2} V^{\alpha}_{\ \ \mu} = \eta_{22} g^{22} V^{2}_{\ \ 2} = (1 + 2\Phi_G)(1 - \Phi_0) \\ V_{33} &= \eta_{\alpha 3} V^{\alpha}_{\ \ 3} = \eta_{33} V^{3}_{\ \ 3} = -(1 - \Phi_0) \\ V_3{}^3 &= \eta_{\alpha 3} g^{\mu 3} V^{\alpha}_{\ \ \mu} = \eta_{33} g^{33} V^{3}_{\ \ 3} = (1 + 2\Phi_G)(1 - \Phi_0) \end{aligned}$$

The remaining vierbein components are of zero value. We will now try to calculate the spin connection values. First, for the case of Γ_0, using equation (4.53), we have,

$$\begin{aligned}
\Gamma_0 =& \frac{1}{2}\sigma^{\alpha\beta}V_\alpha{}^\nu V_{\beta\nu;0} \\
=& \frac{1}{2}\sigma^{\alpha\beta}V_\alpha{}^\nu\left[\partial_0 V_{\beta\nu} - \Gamma^\delta{}_{0\nu}V_{\delta\beta} - \Gamma^\delta{}_{0\beta}V_{\delta\nu}\right] \\
=& -\frac{1}{2}\left[\sigma^{\alpha\beta}V_\alpha{}^\nu\Gamma^\delta{}_{0\nu}V_{\delta\beta} + \sigma^{\alpha\beta}V_\alpha{}^\nu\Gamma^\delta{}_{0\beta}V_{\delta\nu}\right]\ (\partial_0 V_{\beta\nu} = 0) \\
=& -\frac{1}{2}\left[\sigma^{03}V_0{}^0\Gamma^3{}_{00}V_{33} + \sigma^{30}V_3{}^3\Gamma^0{}_{03}V_{00} + \sigma^{30}V_3{}^3\Gamma^3{}_{00}V_{33} + \sigma^{03}V_0{}^0\Gamma^0{}_{03}V_{00}\right] \\
=& -\frac{1}{2}\sigma^{03}\left[V_0{}^0\Gamma^3{}_{00}V_{33} - V_3{}^3\Gamma^0{}_{03}V_{00} - V_3{}^3\Gamma^3{}_{00}V_{33} + V_0{}^0\Gamma^0{}_{03}V_{00}\right] \\
=& -\frac{1}{2}\sigma^{03}\nabla\Phi_G\left[V_0{}^0V_{33} - V_3{}^3V_{00} - V_3{}^3V_{33} + V_0{}^0V_{00}\right]\ (\text{Since } \Gamma^0{}_{03} = \Gamma^3{}_{00} = \nabla\Phi_G) \\
=& -\frac{1}{2}\sigma^{03}\nabla\Phi_G[-(1-2\Phi_G)(1+\Phi_0)(1-\Phi_0) + (1-2\Phi_G)(1+\Phi_0)^2 + (1+2\Phi_0)(1-\Phi_0)^2 \\
& - (1+2\Phi_0)(1-\Phi_0)(1+\Phi_0)] \\
=& -\frac{1}{2}\sigma^{03}\nabla\Phi_G[(1-\Phi_0)(1+\Phi_0)(2\Phi_G - 1 - 1 - 2\Phi_G) + (1-2\Phi_G)\left(1 + 2\Phi_0 + \Phi_0^2\right) \\
& + (1+2\Phi_G)\left(1 - 2\Phi_0 + \Phi_0^2\right)] \\
=& -\frac{1}{2}\sigma^{03}\nabla\Phi_G\left[2 - 2 + 2(\Phi_0)^2 + 2(\Phi_0)^2 - 8\Phi_G\Phi_0\right] \\
\therefore \Gamma_0 =& 2\sigma^{03}\nabla\Phi_G\left[\Phi_G\Phi_0 - (\Phi_0)^2\right]
\end{aligned}$$

Now, using equation (4.54), we have,

$$\sigma^{03} = \frac{1}{4}\left[\gamma_{(0)}{}^0, \gamma_{(0)}{}^3\right] = \frac{1}{4}\left[\gamma_{(0)}{}^0\gamma_{(0)}{}^3 - \gamma_{(0)}{}^3\gamma_{(0)}{}^0\right] = \frac{1}{4}\left[\beta_{(0)}\beta_{(0)}\alpha_{(0)}{}^3 - \beta_{(0)}\alpha_{(0)}{}^3\beta_{(0)}\right]$$

In the next chapter, we will see the following properties of $\beta_{(0)}$ and $\alpha_{(0)}{}^3$,

$$\beta_{(0)}\beta_{(0)} = \mathbb{I} \implies \beta_{(0)}\beta_{(0)}\alpha_{(0)}{}^3 = \alpha_{(0)}{}^3$$

$$\alpha_{(0)}{}^3\beta_{(0)} = -\beta_{(0)}\alpha_{(0)}{}^3 \implies \beta_{(0)}\alpha_{(0)}{}^3\beta_{(0)} = -\beta_{(0)}\beta_{(0)}\alpha_{(0)}{}^3 = -\alpha_{(0)}{}^3$$

$$\therefore \sigma^{03} = \frac{1}{4}\left[\alpha_{(0)}{}^3 - \left(-\alpha_{(0)}{}^3\right)\right] = \frac{\alpha_{(0)}{}^3}{2}$$

Thus, Γ_0 becomes,

$$\Gamma_0 = \alpha_{(0)}{}^3\nabla\Phi_G\left[\Phi_G\Phi_0 - (\Phi_0)^2\right] \tag{4.55}$$

Now for the case of Γ_1,

$$\begin{aligned}
\Gamma_1 =& \frac{1}{2}\sigma^{\alpha\beta}V_\alpha{}^\nu V_{\beta\nu;1}\\
=& \frac{1}{2}\sigma^{\alpha\beta}V_\alpha{}^\nu\left[\frac{\partial V_{\beta\nu}}{\partial x^1} - \Gamma^\delta{}_{1\beta}V_{\delta\nu} - \Gamma^\delta{}_{1\nu}V_{\beta\delta}\right]\\
=& -\frac{1}{2}\left[\sigma^{\alpha\beta}V_\alpha{}^\nu\Gamma^\delta{}_{1\beta}V_{\delta\nu} + \sigma^{\alpha\beta}V_\alpha{}^\nu\Gamma^\delta{}_{1\nu}V_{\beta\delta}\right]\left(\text{Since, } \frac{\partial V_{\beta\nu}}{\partial x^1} = 0)\right)\\
=& -\frac{1}{2}\left[\sigma^{\alpha\beta}V_\alpha{}^\nu\Gamma^1{}_{13}V_{\delta\nu} + \sigma^{\alpha\beta}V_\alpha{}^\nu\Gamma^1{}_{13}V_{\beta\delta} + \sigma^{\alpha\beta}V_\alpha{}^\nu\Gamma^3{}_{11}V_{\delta\nu} + \sigma^{\alpha\beta}V_\alpha{}^\nu\Gamma^3{}_{11}V_{\beta\delta}\right]\\
=& -\frac{1}{2}\left[\sigma^{13}V_1{}^1\Gamma^1{}_{13}V_{11} + \sigma^{31}V_3{}^3\Gamma^1{}_{13}V_{11} + \sigma^{31}V_3{}^3\Gamma^3{}_{11}V_{33} + \sigma^{13}V_1{}^1\Gamma^3{}_{11}V_{33}\right]\\
=& -\frac{1}{2}\sigma^{13}\left[-V_1{}^1\nabla\Phi_G V_{11} + V_3{}^3\nabla\Phi_G V_{11} - V_3{}^3\nabla\Phi_G V_{33} + V_1{}^1\nabla\Phi_G V_{33}\right]\\
=& -\frac{1}{2}\sigma^{13}\nabla\Phi_G\left[-V_1{}^1V_{11} + V_3{}^3V_{11} - V_3{}^3V_{33} + V_1{}^1V_{33}\right]\\
=& -\frac{1}{2}\sigma^{13}\nabla\Phi_G[-(1+2\Phi_G)(1-\Phi_0)^2 + (1+2\Phi_G)(1-\Phi_0)^2\\
&-(1+2\Phi_G)(1-\Phi_0)^2 + (1+2\Phi_G)(1-\Phi_0)^2]\\
\therefore \Gamma_1 =& 0
\end{aligned}$$

For the case of Γ_2, we have,

$$\begin{aligned}
\Gamma_2 =& \frac{1}{2}\sigma^{\alpha\beta}V_\alpha{}^\nu V_{\beta\nu;2}\\
=& \frac{1}{2}\sigma^{\alpha\beta}V_\alpha{}^\nu\left[\frac{\partial V_{\beta\nu}}{\partial x^2} - \Gamma^\delta{}_{2\beta}V_{\delta\nu} - \Gamma^\delta{}_{2\nu}V_{\beta\delta}\right]\\
=& -\frac{1}{2}\left[\sigma^{\alpha\beta}V_\alpha{}^\nu\Gamma^\delta{}_{2\beta}V_{\delta\nu} + \sigma^{\alpha\beta}V_\alpha{}^\nu\Gamma^\delta{}_{2\nu}V_{\beta\delta}\right]\left(\text{Since, } \frac{\partial V_{\beta\nu}}{\partial x^2} = 0\right)\\
=& -\frac{1}{2}\left[\sigma^{\alpha\beta}V_\alpha{}^\nu\Gamma^2{}_{23}V_{\delta\nu} + \sigma^{\alpha\beta}V_\alpha{}^\nu\Gamma^2{}_{23}V_{\beta\delta} + \sigma^{\alpha\beta}V_\alpha{}^\nu\Gamma^3{}_{22}V_{\delta\nu} + \sigma^{\alpha\beta}V_\alpha{}^\nu\Gamma^3{}_{22}V_{\beta\delta}\right]\\
=& -\frac{1}{2}\left[\sigma^{23}V_2{}^2\Gamma^2{}_{23}V_{22} + \sigma^{32}V_3{}^3\Gamma^2{}_{23}V_{22} + \sigma^{32}V_3{}^3\Gamma^3{}_{22}V_{33} + \sigma^{23}V_2{}^2\Gamma^3{}_{22}V_{33}\right]\\
=& -\frac{1}{2}\sigma^{23}\left[V_2{}^2\Gamma^2{}_{23}V_{22} - V_3{}^3\Gamma^2{}_{23}V_{22} - V_3{}^3\Gamma^3{}_{22}V_{33} + V_2{}^2\Gamma^3{}_{22}V_{33}\right]\\
=& -\frac{1}{2}\sigma^{23}\left[-V_2{}^2\nabla\Phi_G V_{22} + V_3{}^3\nabla\Phi_G V_{22} - V_3{}^3\nabla\Phi_G V_{33} + V_2{}^2\nabla\Phi_G V_{33}\right]\\
=& \frac{1}{2}\sigma^{23}\nabla\Phi_G\left[V_2{}^2V_{22} - V_3{}^3V_{22} + V_3{}^3V_{33} - V_2{}^2V_{33}\right]\\
=& \frac{1}{2}\sigma^{23}\nabla\Phi_G[-(1+2\Phi_G)(1-\Phi_0)^2 + (1+2\Phi_G)(1-\Phi_0)^2 - (1+2\Phi_G)(1-\Phi_0)^2\\
&+(1+2\Phi_G)(1-\Phi_0)^2]\\
\therefore \Gamma_2 =& 0
\end{aligned}$$

Finally, we will evaluate Γ_3 as,

$$\begin{aligned}
\Gamma_3 =& \frac{1}{2}\sigma^{\alpha\beta}V_\alpha{}^\nu V_{\beta\nu;3} \\
=& \frac{1}{2}\sigma^{\alpha\beta}V_\alpha{}^\nu\left[\frac{\partial V_{\beta\nu}}{\partial x^3} - \Gamma^\delta{}_{3\beta}V_{\delta\nu} - \Gamma^\delta{}_{3\nu}V_{\beta\delta}\right] \\
=& -\frac{1}{2}\left[\sigma^{\alpha\beta}V_\alpha{}^\nu\Gamma^\delta{}_{3\beta}V_{\delta\nu} + \sigma^{\alpha\beta}V_\alpha{}^\nu\Gamma^\delta{}_{3\nu}V_{\beta\delta}\right]\left(\text{Since, } \frac{\partial V_{\beta\nu}}{\partial x^3} = 0\right) \\
=& -\frac{1}{2}[\sigma^{\alpha\beta}V_\alpha{}^\nu\Gamma^0{}_{03}V_{\delta\nu} + \sigma^{\alpha\beta}V_\alpha{}^\nu\Gamma^0{}_{03}V_{\beta\delta} + \sigma^{\alpha\beta}V_\alpha{}^\nu\Gamma^1{}_{13}V_{\delta\nu} + \sigma^{\alpha\beta}V_\alpha{}^\nu\Gamma^1{}_{13}V_{\beta\delta} \\
&+ \sigma^{\alpha\beta}V_\alpha{}^\nu\Gamma^2{}_{23}V_{\delta\nu} + \sigma^{\alpha\beta}V_\alpha{}^\nu\Gamma^2{}_{23}V_{\beta\delta} + \sigma^{\alpha\beta}V_\alpha{}^\nu\Gamma^3{}_{33}V_{\delta\nu} + \sigma^{\alpha\beta}V_\alpha{}^\nu\Gamma^3{}_{33}V_{\beta\delta}] \\
=& -\frac{1}{2}[\sigma^{00}V_0{}^0\Gamma^0{}_{03}V_{00} + \sigma^{00}V_0{}^0\Gamma^0{}_{03}V_{00} + \sigma^{11}V_1{}^1\Gamma^1{}_{13}V_{11} + \sigma^{11}V_1{}^1\Gamma^1{}_{13}V_{11} \\
&+ \sigma^{22}V_2{}^2\Gamma^2{}_{23}V_{22} + \sigma^{22}V_2{}^2\Gamma^2{}_{23}V_{22} + \sigma^{33}V_3{}^3\Gamma^3{}_{33}V_{33} + \sigma^{33}V_3{}^3\Gamma^3{}_{33}V_{33}]
\end{aligned}$$

Here, using equation (4.54), we see that,

$$\sigma^{00} = \frac{1}{4}\left[\gamma_{(0)}{}^0, \gamma_{(0)}{}^0\right] = \frac{1}{4}\left[\gamma_{(0)}{}^0\gamma_{(0)}{}^0 - \gamma_{(0)}{}^0\gamma_{(0)}{}^0\right] = 0$$

Similarly, σ^{11}, σ^{22} and σ^{33} evaluate to,

$$\begin{aligned}
\sigma^{11} &= \frac{1}{4}\left[\gamma_{(0)}{}^1, \gamma_{(0)}{}^1\right] = \frac{1}{4}\left[\gamma_{(0)}{}^1\gamma_{(0)}{}^1 - \gamma_{(0)}{}^1\gamma_{(0)}{}^1\right] = 0 \\
\sigma^{22} &= \frac{1}{4}\left[\gamma_{(0)}{}^2, \gamma_{(0)}{}^2\right] = \frac{1}{4}\left[\gamma_{(0)}{}^2\gamma_{(0)}{}^2 - \gamma_{(0)}{}^2\gamma_{(0)}{}^2\right] = 0 \\
\sigma^{33} &= \frac{1}{4}\left[\gamma_{(0)}{}^3, \gamma_{(0)}{}^3\right] = \frac{1}{4}\left[\gamma_{(0)}{}^3\gamma_{(0)}{}^3 - \gamma_{(0)}{}^3\gamma_{(0)}{}^3\right] = 0
\end{aligned}$$

As a result, Γ_3 evaluates to,

$$\therefore \Gamma_3 = 0$$

Thus, we have obtained the spin connection values to be,

$$\Gamma_0 = \alpha_{(0)}{}^3\nabla\Phi_G\left[\Phi_G\Phi_0 - (\Phi_0)^2\right] \tag{4.56}$$

$$\Gamma_1 = \Gamma_2 = \Gamma_3 = 0 \tag{4.57}$$

In the equation (4.56), we see that it is of the second-order, either in Φ_G or in Φ_0 and the equation (4.57) tell us that the spatial spin connections are all zero. Hence, we can say that the spin connection can be safely neglected in the Dirac equation for an electron in a spherically static and symmetric gravitational field.

Chapter 5

Calculation of electron acceleration using Dirac equation

Here, we will try to obtain the acceleration of an electron in a uniform gravitational field, such that it is moving along the z-direction, with a velocity v, that is small compared to the speed of light, but large enough to consider terms up to the second order of v/c.

5.1 Modification of Dirac equation

As discussed before, the Dirac equation in curved spacetime is given by,

$$i\gamma^{\mu}(z)\mathscr{D}_{\mu}\psi(z) = m\psi(z) \tag{5.1}$$

where,

$$\mathscr{D}_{\mu} = \partial_{\mu} + \Gamma_{\mu}$$

The Dirac matrices in flat spacetime ${\gamma_{(0)}}^{\mu}(z)$, and in curved spacetime $\gamma^{\mu}(z)$, are related as follows,

$$\gamma^{0}(z) = [1 - \Phi_G(z)]{\gamma_{(0)}}^{0}(z) \tag{5.2}$$

$$\gamma^{i}(z) = [1 + \Phi_G(z)]{\gamma_{(0)}}^{i}(z) \tag{5.3}$$

We now define a new set of matrices $\boldsymbol{\alpha}_{(0)}$ and $\beta_{(0)}$, as follows,

$$\beta_{(0)} = {\gamma_{(0)}}^{0} \tag{5.4}$$

$$\beta_{(0)}{\alpha_{(0)}}^{i} = {\gamma_{(0)}}^{i} \tag{5.5}$$

Substituting equation (5.4) into equation (5.2) and equation (5.5) into equation (5.3), we obtain the following relations,

$$\gamma^{0}(z) = [1 - \Phi_G]\beta_{(0)}$$

$$\gamma^{i}(z) = [1 + \Phi_G]\beta_{(0)}{\alpha_0}^{i}$$

Now, equation (5.1), after substituting the expression for $\mathscr{D}_{\mu}$ (after neglecting the spin connection term, for reasons shown in the previous chapter), becomes,

$$i\gamma^{\mu}(x)\partial_{\mu}\psi(x) = m\psi(x) \tag{5.6}$$

This equation can be modified as follows,

$$\begin{aligned}
&i\gamma^0(x)\partial_0\psi(x) + i\gamma^i(x)\partial_i\psi(x) = m\psi(x)\\
\implies &i\gamma^0(x)\frac{\partial\psi}{\partial t} = -i\gamma^i\nabla\psi(x) + m\psi(x)\\
\implies &i[1-\Phi_G]\beta_{(0)}\frac{\partial\psi}{\partial t} = [1+\Phi_G]\beta_{(0)}\boldsymbol{\alpha}_{(0)}\cdot\left(\frac{\boldsymbol{\nabla}}{i}\right)\psi + m\psi
\end{aligned}$$

The above is obtained by substituting the values of $\gamma^0(z)$ and $\gamma^i(z)$ we had previously obtained. Dividing the above equation throughout by $[1-\Phi_G]$, we get,

$$\begin{aligned}
i\beta_{(0)}\frac{\partial\psi}{\partial t} &= \left(\frac{1+\Phi_G}{1-\Phi_G}\right)\beta_{(0)}\boldsymbol{\alpha}_{(0)}\cdot\left(\frac{\boldsymbol{\nabla}}{i}\right)\psi + \frac{m}{1-\Phi_G}\psi\\
&= (1+\Phi_G)(1-\Phi_G)^{-1}\beta_{(0)}\boldsymbol{\alpha}_{(0)}\cdot\left(\frac{\boldsymbol{\nabla}}{i}\right)\psi + m(1-\Phi_G)^{-1}\psi\\
&\simeq [1+\Phi_G][1+\Phi_G]\beta_{(0)}\boldsymbol{\alpha}_{(0)}\cdot\left(\frac{\boldsymbol{\nabla}}{i}\right)\psi + m(1+\Phi_G)\psi\\
&\simeq [1+2\Phi_G]\beta_{(0)}\boldsymbol{\alpha}_{(0)}\cdot\left(\frac{\boldsymbol{\nabla}}{i}\right)\psi + m(1+\Phi_G)\psi
\end{aligned}$$

In the above expressions, we have considered only the first order of Φ_G, neglecting the higher orders of Φ_G.

$$\therefore i\beta_{(0)}\frac{\partial\psi}{\partial t} = [1+2\Phi_G]\beta_{(0)}\boldsymbol{\alpha}_{(0)}\cdot\left(\frac{\boldsymbol{\nabla}}{i}\right)\psi + m(1+\Phi_G)\psi \tag{5.7}$$

Now, we have,

$$\beta_{(0)} = {\gamma_{(0)}}^0 = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The inverse of the above matrix will be,

$$\left(\beta_{(0)}\right)^{-1} = \left({\gamma_{(0)}}^0\right)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Premultiplying equation (5.7) by $(\beta_{(0)})^{-1}$, we get,

$$i\left(\beta_{(0)}\right)^{-1}\beta_{(0)}\frac{\partial\psi}{\partial t} = [1+2\Phi_G]\left(\beta_{(0)}\right)^{-1}\beta_{(0)}\boldsymbol{\alpha}_{(0)}\cdot\left(\frac{\boldsymbol{\nabla}}{i}\right)\psi + m(1+\Phi_G)\left(\beta_{(0)}\right)^{-1}\psi$$

$$\therefore i\frac{\partial\psi}{\partial t} = [1+2\Phi_G]\boldsymbol{\alpha}_{(0)}\cdot\left(\frac{\boldsymbol{\nabla}}{i}\right)\psi + m(1+\Phi_G)\beta_{(0)}\psi \tag{5.8}$$

Here, $\beta_{(0)} = \left(\beta_{(0)}\right)^{-1}$.The equation (5.8) represents the Dirac equation in curved spacetime, upto the first order of Φ_G. The last term in the above equation consist of $\beta_{(0)}$ which is simply a diagonal matrix, consisting of $1, 1, -1$ and -1 as its diagonal elements, and hence the last term corresponds to the gravitational potential of the electron.

It can also be said that the gravitational effects show up in the Dirac equation by a metric rather than by directly including them in the Hamiltonian. One consequence of this is the appearance of a factor of $(1+2\Phi_G)$ in the first term in equation (5.8), which represents the kinetic energy.

5.2 Transformation of spin in flat spacetime

Spin is considered to be a type of intrinsic angular momentum, and thus, it will have transformation properties similar to angular momentum. The treatment in this section will be considered to be in flat spacetime only. The expression for angular momentum is given by,

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$$

In tensor notation, this can be written as,

$$L_i = \epsilon_{ijk} r_j p_k$$

Now, $\boldsymbol{L}$ being the product of two vectors, it must show the transformation properties of a second rank tensor. So let us see how a second rank tensor undergoes a Lorentz transformation. For this, we will make use of the electromagnetic field tensor as an example of our second-rank tensor. It is defined as follows,

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & -B_3 & B_2 \\ E_2/c & B_3 & 0 & -B_1 \\ E_3/c & -B_2 & B_1 & 0 \end{pmatrix} \tag{5.9}$$

Under a Lorentz transformation, the components of the electric and the magnetic field, that is parallel and perpendicular to the velocity vector, transforms as follows,

$$B_{\parallel}{}' = B_{\parallel}, \; E_{\parallel}{}' = E_{\parallel} \tag{5.10}$$

$$\boldsymbol{B}_{\perp}{}' = \frac{\boldsymbol{B}_{\perp} - (\boldsymbol{v}/c) \times \boldsymbol{E}}{\sqrt{1 - v^2/c^2}}, \; \boldsymbol{E}_{\perp}{}' = \frac{\boldsymbol{E}_{\perp} + (\boldsymbol{v}/c) \times \boldsymbol{B}}{\sqrt{1 - v^2/c^2}} \tag{5.11}$$

Here, $E_{\perp}$ and $B_{\perp}$ refer to the components of electric and magnetic field, that is perpendicular to the velocity vector, respectively, and $E_{\parallel}$ and $B_{\parallel}$ refer to the components of electric and magnetic field, that is parallel to the velocity vector, respectively. Also, the primed and the unprimed coordinates are measured in the two inertial frames of reference, moving relative to each other.

Now, under spatial inversion, the electric field shows odd parity ($\boldsymbol{E} \to -\boldsymbol{E}$), while both the angular momentum and the magnetic field show even parity ($\boldsymbol{B} \to \boldsymbol{B}$, $\boldsymbol{L} \to \boldsymbol{L}$). By analogy, we can therefore say that the angular momentum (and spin) should show similar transformation properties as the magnetic field, $\boldsymbol{B}$. For the case of spin $\boldsymbol{S} = \boldsymbol{\sigma}/2$, there will be another set of dynamical variables that represents the internal degrees of freedom of the particle. Let these variables be $i\boldsymbol{\alpha}/2$ (with $i/2$ for convenience), and will transform analogous to $\boldsymbol{E}$. Thus, the transformations of $\boldsymbol{S}$ (or $\boldsymbol{\sigma}$) and $i\boldsymbol{\alpha}/2$, as per equation (5.10) and equation (5.11) are as follows,

$$\sigma_{\parallel}{}' = \sigma_{\parallel}, \; i\alpha_{\parallel}{}' = i\alpha_{\parallel} \tag{5.12}$$

$$\boldsymbol{\sigma}_{\perp}{}' = \frac{\boldsymbol{\sigma}_{\perp} - (\boldsymbol{v}/c) \times i\boldsymbol{\alpha}}{\sqrt{1 - v^2/c^2}}, \; i\boldsymbol{\alpha}_{\perp}{}' = \frac{i\boldsymbol{\alpha}_{\perp} + (\boldsymbol{v}/c) \times \boldsymbol{\sigma}}{\sqrt{1 - v^2/c^2}} \tag{5.13}$$

Here also, the symbols $\parallel$ and $\perp$ refer to the components of $\boldsymbol{\sigma}$ and $i\boldsymbol{\alpha}$ that are parallel and perpendicular to the velocity vector, respectively. Now, the second-order matrix that is analogous to the Maxwell field strength tensor $F^{\mu\nu}$, is as follows,

$$\sigma^{\mu\nu} = \begin{pmatrix} 0 & -i\alpha_1 & -i\alpha_2 & -i\alpha_3 \\ i\alpha_1 & 0 & \sigma_3 & -\sigma_2 \\ i\alpha_2 & -\sigma_3 & 0 & \sigma_1 \\ i\alpha_3 & \sigma_2 & -\sigma_1 & 0 \end{pmatrix} \tag{5.14}$$

The algebraic properties of $\boldsymbol{\alpha}$ can be found with the help of the algebraic properties of spin, and also the fact that the spin generates rotations of internal degrees of freedom. As $\boldsymbol{\alpha}$ behaves like a vector under spatial rotations, it must commute with $\boldsymbol{S}$ (or $\boldsymbol{\sigma}$) as follows,

$$[\alpha_i, S_j] = \alpha_i S_j - S_j \alpha_i = i\epsilon_{ijk}\alpha_k \tag{5.15}$$

$$[\alpha_i, \sigma_j] = \alpha_i \sigma_j - \sigma_j \alpha_i = 2i\epsilon_{ijk}\alpha_k \tag{5.16}$$

Equation (5.15) and equation (5.16) is valid for any spin values except for spin zero, in which case, $\boldsymbol{\alpha} = \mathbb{O}$. For the case of spin-1/2, $\boldsymbol{\sigma}$ obeys the following relation,

$$\sigma_i \sigma_j = i\epsilon_{ijk}\sigma_k + \delta_{ij} \tag{5.17}$$

Here, ϵ_{ijk} is the Levi-Civita symbol and δ_{ij} is the Kronecker delta. The equation (5.17) should be true for all spin operators in any Lorentz frame.

Suppose we take $\boldsymbol{v}$ to be along the z-axis. This can also be written as,

$$\boldsymbol{v} = v\hat{\mathbf{z}}$$

in which, $\hat{\mathbf{z}}$ is the unit vector along the z-axis. Now, in equation (5.17), let us consider the case of $i = j$.

$$(\sigma_i)^2 = \sigma_i \sigma_i = i\epsilon_{iik}\sigma_k + \delta_{ii}$$
$$\therefore (\sigma_i)^2 = 1 \implies \sigma_{x'} = 1$$

Also, from equation (5.13), we have,

$$\sigma_{x'} = \frac{\sigma_x - [(\boldsymbol{v}/c) \times i\boldsymbol{\alpha}]_x}{\sqrt{1 - v^2/c^2}} \tag{5.18}$$

$$\sigma_{y'} = \frac{\sigma_y - [(\boldsymbol{v}/c) \times i\boldsymbol{\alpha}]_y}{\sqrt{1 - v^2/c^2}} \tag{5.19}$$

Here, the subscripts x and y represents the $x-$ and $y-$ components of the cross-product of $\boldsymbol{v}/c$ and $i\boldsymbol{\alpha}$. Now,

$$\boldsymbol{\alpha} = \alpha_x\hat{\mathbf{x}} + \alpha_y\hat{\mathbf{y}} + \alpha_z\hat{\mathbf{z}}$$

where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ refer to the unit vectors along the $x-$, $y-$ and $z-$ axes respectively.

$$\therefore (\boldsymbol{v}/c) \times (i\boldsymbol{\alpha}) = \frac{i}{c}[(v\hat{\mathbf{z}}) \times (\alpha_x\hat{\mathbf{x}} + \alpha_y\hat{\mathbf{y}} + \alpha_z\hat{\mathbf{z}})]$$

$$\boldsymbol{v} \times \boldsymbol{\alpha} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & v \\ \alpha_x & \alpha_y & \alpha_z \end{vmatrix} = (-v\alpha_y)\hat{\mathbf{x}} + (\alpha_x v)\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$$

$$\therefore [(\boldsymbol{v}/c) \times (i\boldsymbol{\alpha})]_x = -iv\alpha_y/c, \; [(\boldsymbol{v}/c) \times (i\boldsymbol{\alpha})]_y = iv\alpha_x/c$$

Thus, equation (5.18) and equation (5.19) become,

$$\sigma_{x'} = \frac{\sigma_x + iv\alpha_y/c}{\sqrt{1 - v^2/c^2}} \tag{5.20}$$

$$\sigma_{y'} = \frac{\sigma_y - iv\alpha_x/c}{\sqrt{1 - v^2/c^2}} \tag{5.21}$$

Squaring the expression for $\sigma_{x'}$ found in the equation (5.20), we get,

$$\begin{aligned}(\sigma_{x'})^2 &= \frac{(\sigma_x + iv\alpha_y/c)^2}{1 - v^2/c^2} \\ &= \frac{(\sigma_x)^2 + iv\sigma_x(\alpha_y/c) + iv(\alpha_y/c)\sigma_x - (v/c)^2(\alpha_y)^2}{1 - v^2/c^2}\end{aligned}$$

Since $(\sigma_{x'})^2 = (\sigma_x)^2 = 1$, we have,

$$\frac{1 + i(v/c)(\sigma_x\alpha_y + \alpha_y\sigma_x) - (v/c)^2(\alpha_y)^2}{1 - v^2/c^2} = 1$$

For the above equation to hold, we can deduce the following,

$$(\alpha_y)^2 = 1$$

$$\sigma_x\alpha_y + \alpha_y\sigma_x = 0$$

In general, this can be written as,

$$\sigma_i\alpha_j + \alpha_j\sigma_i = 0 \text{ (for } i \neq j)$$

Now, multiplying both sides of equation (5.21) by $\sigma_{z'}$, we get,

$$\sigma_{y'}\sigma_{z'} = \frac{\sigma_y\sigma_{z'} - iv\alpha_x\sigma_{z'}/c}{\sqrt{1 - v^2/c^2}} \tag{5.22}$$

Since the parallel component is along the $z-$axis, we have,

$$\sigma_{z'} = \sigma_z$$

Also, we can use the following equations from equation (5.17),

$$\sigma_{y'}\sigma_{z'} = i\sigma_{x'}, \;\; \sigma_y\sigma_z = i\sigma_x$$

Thus, equation (5.22) becomes,

$$i\sigma_{x'} = \frac{i\sigma_x - iv\alpha_x\sigma_{z'}/c}{\sqrt{1 - v^2/c^2}} \implies \sigma_{x'} = \frac{\sigma_x - v\alpha_x\sigma_z/c}{\sqrt{1 - v^2/c^2}}$$

Comparing the above equation with equation (5.20), we obtain the following relation,

$$\begin{aligned}\frac{\sigma_x + iv\alpha_y/c}{\sqrt{1 - v^2/c^2}} &= \frac{\sigma_x - v\alpha_x\sigma_z/c}{\sqrt{1 - v^2/c^2}} \\ i\alpha_y &= -\alpha_x\sigma_z \\ \therefore \alpha_x\sigma_z &= -i\alpha_y \end{aligned} \tag{5.23}$$

Now, multiplying $\sigma_{z'}$ on both sides of equation (5.20),

$$\sigma_{x'}\sigma_{z'} = \frac{\sigma_x\sigma_{z'} + iv\alpha_y\sigma_{z'}/c}{\sqrt{1 - v^2/c^2}} \tag{5.24}$$

Here also, since $z-$axis is the direction of the parallel component, we have,

$$\sigma_{z'} = \sigma_z$$

and also, using equation (5.17),

$$\sigma_{x'}\sigma_{z'} = -i\sigma_{y'},\ \sigma_x\sigma_z = -i\sigma_y$$

Thus, equation (5.24) modifies to,

$$-i\sigma_{y'} = \frac{-i\sigma_y + iv\alpha_y\sigma_{z'}/c}{\sqrt{1-v^2/c^2}} \implies -\sigma_{y'} = \frac{-\sigma_y + v\alpha_y\sigma_z/c}{\sqrt{1-v^2/c^2}}$$

Comparing the above equation with equation (5.21), we get,

$$\frac{-\sigma_y + iv\alpha_x/c}{\sqrt{1-v^2/c^2}} = \frac{-\sigma_y + v\alpha_y\sigma_z/c}{\sqrt{1-v^2/c^2}}$$

$$i\alpha_x = \alpha_y\sigma_z$$

$$\therefore \alpha_y\sigma_z = i\alpha_x \tag{5.25}$$

From equation (5.23) and equation (5.25), we can write in general,

$$\alpha_i\sigma_j = i\epsilon_{ijk}\alpha_k \text{ (for } i \neq j)$$

Now multiplying equation (5.20) and equation (5.21), we obtain the following,

$$\sigma_{x'}\sigma_{y'} = \frac{(\sigma_x + iv\alpha_y/c)(\sigma_y - iv\alpha_x/c)}{1-v^2/c^2}$$

Now, from equation (5.17) and since $z-$axis is the direction of the parallel component, we have $\sigma_{x'}\sigma_{y'} = i\sigma_{z'} = i\sigma_z$, and the above expression modifies to,

$$\begin{aligned} i\sigma_{z'} &= \frac{\sigma_x\sigma_y - iv\sigma_x\alpha_x/c + iv\alpha_y\sigma_y/c + (v/c)^2\alpha_y\alpha_x}{1-v^2/c^2} \\ &= \frac{i\sigma_z + (iv/c)(\alpha_y\sigma_y - \sigma_x\alpha_x) + (v/c)^2\alpha_y\alpha_x}{1-v^2/c^2} \\ \therefore i\sigma_z\big(1-v^2/c^2\big) &= i\sigma_z + (iv/c)(\alpha_y\sigma_y - \sigma_x\alpha_x) + v^2/c^2\alpha_y\alpha_x \end{aligned}$$

Comparing the coefficients of v^2/c^2, on both sides of the equation (for the above equation to be valid), we get,

$$\alpha_y\alpha_z = -i\sigma_z \tag{5.26}$$

Now, when we multiply equation (5.21) and equation (5.20) in order, we have,

$$\begin{aligned} \sigma_{y'}\sigma_{x'} &= \frac{(\sigma_y - iv\alpha_x/c)(\sigma_x + iv\alpha_y/c)}{1-v^2/c^2} \\ &= \frac{\sigma_y\sigma_x + i(v/c)\sigma_y\alpha_y - i(v/c)\alpha_x\sigma_x + (v/c)^2\alpha_x\alpha_y}{1-v^2/c^2} \\ \therefore \sigma_{y'}\sigma_{x'}\big(1-v^2/c^2\big) &= \sigma_y\sigma_x + i(v/c)(\sigma_y\alpha_y - \alpha_x\sigma_x) + (v/c)^2\alpha_x\alpha_y \end{aligned}$$

Once again, from equation (5.17) and since $z-$axis is the direction of the parallel component, we have $\sigma_{y'}\sigma_{x'} = -i\sigma_{z'} = -i\sigma_z$ and $\sigma_y\sigma_x = -i\sigma_z$, and the above expression becomes,

$$-(i\sigma_z) + (i\sigma_z)(v/c)^2 = -(i\sigma_z) + i(v/c)(\sigma_y\alpha_y - \alpha_x\sigma_x) + (v/c)^2\alpha_x\alpha_y$$

Comparing the coefficients of $(v/c)^2$ on both sides of the equation, we get,

$$\alpha_x\alpha_y = i\sigma_z \tag{5.27}$$

Adding equation (5.26) and equation (5.27), we get,

$$\alpha_x\alpha_y + \alpha_y\alpha_x = 0$$

In general, this can be written in the form of the anti-commutator of different components of $\boldsymbol{\alpha}$,

$$\{\alpha_i, \alpha_j\} = \alpha_i\alpha_j + \alpha_j\alpha_i = 0 \tag{5.28}$$

with the condition that $i \neq j$. If we subtract equation (5.27) from equation (5.26),

$$\alpha_x\alpha_y - \alpha_y\alpha_x = 2i\sigma_z$$

which can be generally written in the form of the commutator of the components of $\boldsymbol{\alpha}$ as,

$$[\alpha_i, \alpha_j] = \alpha_i\alpha_j - \alpha_j\alpha_i = 2i\epsilon_{ijk}\sigma_k \tag{5.29}$$

Under a parity operation, $\boldsymbol{\sigma}$ and $\boldsymbol{\alpha}$ transform as follows,

$$\boldsymbol{\sigma} \to \boldsymbol{\sigma};\ \boldsymbol{\alpha} \to -\boldsymbol{\alpha}$$

This transformation is similar to to how electric and magnetic field change under parity operation ($\boldsymbol{E} \to -\boldsymbol{E}$ and $\boldsymbol{B} \to \boldsymbol{B}$), and thus, we can see that the space-space components of their respective rank-2 tensor ($F^{\mu\nu}$ and $\sigma^{\mu\nu}$) do not change signs under parity whereas the time-space components change signs.

We will now consider a matrix β operator that performs the parity operation in spin space. Since two inversions bring it back to the initial coordinates, we need it to have $\beta^2 = \mathbb{I}$. However, since the spin representation of rotations can have two values, we can therefore, have an option letting the square of the parity operator include a rotation by 2π about any axis or not, in which case, we have $\beta^2 = \pm\mathbb{I}$. The resulting eigenvalues are $\beta = \pm\mathbb{I}$ for $\beta^2 = \mathbb{I}$ and $\beta = \pm i\mathbb{I}$ for $\beta^2 = -\mathbb{I}$. For convenience's sake, let us choose,

$$\beta^2 = \mathbb{I} \implies \beta^{-1} = \beta$$

Now under a parity operation, we have said that $\boldsymbol{\sigma}$ does not change sign while $\boldsymbol{\alpha}$ changes sign,

$$\beta^{-1}\boldsymbol{\sigma}\beta = \boldsymbol{\sigma} \implies \boldsymbol{\sigma}\beta = \beta\boldsymbol{\sigma}$$
$$\beta^{-1}\boldsymbol{\alpha}\beta = -\boldsymbol{\alpha} \implies \boldsymbol{\alpha}\beta = -\beta\boldsymbol{\alpha}$$

5.3 Iterative calculation of electron acceleration

We had earlier obtained the Dirac equation in curved spacetime, as per equation (5.8), as,

$$i\frac{\partial\psi}{\partial t} = (1 + 2\Phi_G)\boldsymbol{\alpha}_{(0)} \cdot \left(\frac{\boldsymbol{\nabla}}{i}\right)\psi + m(1 + \Phi_G)\beta_{(0)}\psi \tag{5.30}$$

Here,

$$\boldsymbol{\alpha}_{(0)} = \begin{pmatrix} 0 & \boldsymbol{\tau} \\ \boldsymbol{\tau} & 0 \end{pmatrix},\ \beta_{(0)} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} \tag{5.31}$$

The wave function ψ is a four-component spinor which can be written in terms of two two-component spinors ϕ and χ, as follows,

$$\psi(x^\mu) = \begin{bmatrix} \phi(x^\mu) \\ \chi(x^\mu) \end{bmatrix} \tag{5.32}$$

Substituting equation (5.31) and equation (5.32) into equation (5.30), we get,

$$\begin{aligned}
i\frac{\partial}{\partial t}\begin{bmatrix}\phi(x^\mu)\\ \chi(x^\mu)\end{bmatrix} &= (1+2\Phi_G)\begin{bmatrix}0 & \boldsymbol{\tau}\\ \boldsymbol{\tau} & 0\end{bmatrix}\cdot\left(\frac{\boldsymbol{\nabla}}{i}\right)\begin{bmatrix}\phi(x^\mu)\\ \chi(x^\mu)\end{bmatrix} + (1+\Phi_G)\begin{bmatrix}\mathbb{I} & 0\\ 0 & -\mathbb{I}\end{bmatrix} m\begin{bmatrix}\phi(x^\mu)\\ \chi(x^\mu)\end{bmatrix}\\
&= (1+2\Phi_G)\begin{bmatrix}\boldsymbol{\tau}\cdot(\boldsymbol{\nabla}/i)\chi(x^\mu)\\ \boldsymbol{\tau}\cdot(\boldsymbol{\nabla}/i)\phi(x^\mu)\end{bmatrix} + (1+\Phi_G)\begin{bmatrix}m\phi(x^\mu)\\ -m\chi(x^\mu)\end{bmatrix}\\
\therefore i\frac{\partial}{\partial t}\begin{bmatrix}\phi(x^\mu)\\ \chi(x^\mu)\end{bmatrix} &= (1+2\Phi_G)\begin{bmatrix}\boldsymbol{\tau}\cdot(\boldsymbol{\nabla}/i)\chi(x^\mu)\\ \boldsymbol{\tau}\cdot(\boldsymbol{\nabla}/i)\phi(x^\mu)\end{bmatrix} + (1+\Phi_G)\begin{bmatrix}m\phi(x^\mu)\\ -m\chi(x^\mu)\end{bmatrix}
\end{aligned}$$

Comparing the matrix elements, we can get the following equations,

$$i\frac{\partial(\phi(x^\mu))}{\partial t} = (1+2\Phi_G)\boldsymbol{\tau}\cdot\left(\frac{\boldsymbol{\nabla}}{i}\right)\chi(x^\mu) + (1+\Phi_G)m\phi(x^\mu) \tag{5.33}$$

$$i\frac{\partial(\chi(x^\mu))}{\partial t} = (1+2\Phi_G)\boldsymbol{\tau}\cdot\left(\frac{\boldsymbol{\nabla}}{i}\right)\phi(x^\mu) - (1+\Phi_G)m\chi(x^\mu) \tag{5.34}$$

Equation (5.33) and equation (5.34) can also be used in the case of special relativity, in which case, $\Phi_G = 0$. If we consider only the rest-mass energy of the particle, then we can approximately write the two-component spinor as,

$$\begin{bmatrix}\phi(x^\mu)\\ \chi(x^\mu)\end{bmatrix} = \begin{bmatrix}\phi_0(x^\mu)\\ \chi_0(x^\mu)\end{bmatrix}\mathrm{e}^{-imt}$$

where, $\phi_0(x^\mu)$ and $\chi_0(x^\mu)$ are slowly-varying functions of spacetime that represent the inital condition of the two-component spinor wave function, and the exponential term e^{-imt} is time-evolution operator. Do note that in the above equation, we have used the natural units in which $c = \hbar = 1$. This equation can also be written as,

$$\phi(x^\mu) = \phi_0(x^\mu)\mathrm{e}^{-imt} \tag{5.35}$$

$$\chi(x^\mu) = \chi_0(x^\mu)\mathrm{e}^{-imt} \tag{5.36}$$

Substituting equation (5.35) and equation (5.36) into equation (5.34).

$$\begin{aligned}
i\frac{\partial}{\partial t}\left[\chi_0(x^\mu)\mathrm{e}^{-imt}\right] &= (1+2\Phi_G)\boldsymbol{\tau}\cdot\left(\frac{\boldsymbol{\nabla}}{i}\right)\phi_0(x^\mu)\mathrm{e}^{-imt} - (1+\Phi_G)m\chi_0(x^\mu)\mathrm{e}^{-imt}\\
i\left[\mathrm{e}^{-imt}\right]\left[\frac{\partial\chi_0}{\partial t} - i\chi_0 m\right] &= (1+2\Phi_G)\boldsymbol{\tau}\cdot\left(\frac{\boldsymbol{\nabla}}{i}\right)\phi_0\mathrm{e}^{-imt} - (1+\Phi_G)m\chi_0\mathrm{e}^{-imt}\\
\therefore i\frac{\partial\chi_0}{\partial t} &= (1+2\Phi_G)\boldsymbol{\tau}\cdot\left(\frac{\boldsymbol{\nabla}}{i}\right)\phi_0 - (\Phi_G+2)m\chi_0 \qquad (5.37)
\end{aligned}$$

Now, in the non-relativistic limit, where the velocity of the electron is small compared to the speed of light, it is convenient to assume that the energy and the gravitational potential energy of the particle is small compared to the rest mass-energy of the particle, that is,

$$i\frac{\partial\chi_0}{\partial t} \ll |2m\chi_0|,\ |m\Phi_G\chi_0| \ll |2m\chi_0|$$

Thus, equation (5.37) can be modified as,

$$(1+2\Phi_G)\boldsymbol{\tau}\cdot\left(\frac{\boldsymbol{\nabla}}{i}\right)\phi_0 - 2m\chi_0 = 0$$

This equation can be modified as,

$$\chi_0 = \left(\frac{1+2\Phi_G}{2m}\right)\left[\boldsymbol{\tau}\cdot\left(\frac{\boldsymbol{\nabla}}{i}\right)\right]\phi_0$$

Multiplying both sides of the above equation with e^{-imt}, we get,

$$\chi = \left(\frac{1+2\Phi_G}{2m}\right)\left[\boldsymbol{\tau}\cdot\left(\frac{\boldsymbol{\nabla}}{i}\right)\right]\phi \tag{5.38}$$

Here, χ is referred to as the small component of the wave function ψ, while ϕ is referred to as the large component of the wave function ψ.

Now, we will make use of an iterative procedure that makes use of the equation (5.33), the equation (5.34), and the equation (5.38), that results in an equation that is similar to the non-relativistic Schrödinger equation. For that, let us re-write equation (5.34) as follows,

$$\begin{aligned}
i\frac{\partial\chi}{\partial t} &= (1+2\Phi_G)\left[\boldsymbol{\tau}\cdot\left(\frac{\boldsymbol{\nabla}}{i}\right)\right]\phi - (1+2\Phi_G)m\chi \\
&= (1+2\Phi_G)\left[\boldsymbol{\tau}\cdot\left(\frac{\boldsymbol{\nabla}}{i}\right)\right]\phi - 2m\chi + m\chi - \Phi_G m\chi \\
\therefore 2m\chi &= (1+2\Phi_G)\left[\boldsymbol{\tau}\cdot\left(\frac{\boldsymbol{\nabla}}{i}\right)\right]\phi - i\frac{\partial\chi}{\partial t} + m\chi - \Phi_G m\chi \\
\therefore \chi &= \frac{(1+2\Phi_G)}{2m}\left[\boldsymbol{\tau}\cdot\left(\frac{\boldsymbol{\nabla}}{i}\right)\right]\phi - \frac{1}{2m}\left[i\frac{\partial}{\partial t} - m + m\Phi_G\right]\chi
\end{aligned}$$

Now, we substitute the χ value obtained in equation (5.38) into the right-hand side of the above equation, and obtain,

$$\chi = \frac{(1+2\Phi_G)}{2m}\left[\boldsymbol{\tau}\cdot\left(\frac{\boldsymbol{\nabla}}{i}\right)\right]\phi - \frac{1}{2m}\left[i\frac{\partial}{\partial t} - m + m\Phi_G\right]\left[\frac{(1+2\Phi_G)}{2m}\right]\left[\boldsymbol{\tau}\cdot\left(\frac{\boldsymbol{\nabla}}{i}\right)\right]\phi$$

Substituting the above value of χ into equation (5.33), we obtain,

$$\begin{aligned}
i\frac{\partial\phi}{\partial t} =&(1+2\Phi_G)\left[\boldsymbol{\tau}\cdot\left(\frac{\boldsymbol{\nabla}}{i}\right)\right]\left[\left(\frac{1+2\Phi_G}{2m}\right)\left[\boldsymbol{\tau}\cdot\left(\frac{\boldsymbol{\nabla}}{i}\right)\right]\phi\right. \\
&\left. - \frac{1}{2m}\left(i\frac{\partial}{\partial t} - m + m\Phi_G\right)\left(\frac{1+2\Phi_G}{2m}\right)\left[\boldsymbol{\tau}\cdot\left(\frac{\boldsymbol{\nabla}}{i}\right)\right]\right]\phi + (1+\Phi_G)m\phi
\end{aligned}$$

Writing $\left(\frac{\boldsymbol{\nabla}}{i}\right)$ as $\hat{\mathbf{p}}$, the above equation becomes,

$$\begin{aligned}
i\frac{\partial\phi}{\partial t} =&(1+2\Phi_G)(\boldsymbol{\tau}\cdot\hat{\mathbf{p}})\left[\left(\frac{1+2\Phi_G}{2m}\right)(\boldsymbol{\tau}\cdot\hat{\mathbf{p}})\phi\right. \\
&\left. - \frac{1}{2m}\left(i\frac{\partial}{\partial t} - m + m\Phi_G\right)\left(\frac{1+2\Phi_G}{2m}\right)(\boldsymbol{\tau}\cdot\hat{\mathbf{p}})\right]\phi + (1+\Phi_G)m\phi \\
\therefore i\frac{\partial\phi}{\partial t} =&\left[(1+2\Phi_G)(\boldsymbol{\tau}\cdot\hat{\mathbf{p}})\left(\frac{1+2\Phi_G}{2m}\right)(\boldsymbol{\tau}\cdot\hat{\mathbf{p}})\right]\phi + m\phi + m\Phi_G\phi \\
&- \left[\left(\frac{1+2\Phi_G}{2m}\right)(\boldsymbol{\tau}\cdot\hat{\mathbf{p}})\left(i\frac{\partial}{\partial t} - m + m\Phi_G\right)\left(\frac{1+2\Phi_G}{2m}\right)(\boldsymbol{\tau}\cdot\hat{\mathbf{p}})\right]\phi \\
=&\left[\boldsymbol{\tau}\cdot(1+2\Phi_G)\hat{\mathbf{p}}\right]\left[\boldsymbol{\tau}\cdot\left(\frac{1+2\Phi_G}{2m}\right)\hat{\mathbf{p}}\right]\phi + m\phi + m\Phi_G\phi \\
&- \left[\boldsymbol{\tau}\cdot(1+2\Phi_G)\hat{\mathbf{p}}\right]\left[\boldsymbol{\tau}\cdot\frac{1}{2m}\left(i\frac{\partial}{\partial t} - m + m\Phi_G\right)\left(\frac{1+2\Phi_G}{2m}\right)\hat{\mathbf{p}}\right]\phi
\end{aligned}$$

We will now make use of the following property of $\boldsymbol{\tau}$,

$$(\boldsymbol{\tau}\cdot\boldsymbol{A})(\boldsymbol{\tau}\cdot\boldsymbol{B}) = \boldsymbol{A}\cdot\boldsymbol{B} + i\boldsymbol{\tau}\cdot(\boldsymbol{A}\times\boldsymbol{B})$$

and the above equations become,

$$\begin{aligned}
i\frac{\partial\phi}{\partial t} =& \left[(1+2\Phi_G)\hat{\mathbf{p}}\cdot\left(\frac{1+2\Phi_G}{2m}\right)\hat{\mathbf{p}}\right]\phi + i\boldsymbol{\tau}\cdot\left[(1+2\Phi_G)\hat{\mathbf{p}}\times\left(\frac{1+2\Phi_G}{2m}\right)\hat{\mathbf{p}}\right]\phi \\
&- \left[(1+2\Phi_G)\hat{\mathbf{p}}\cdot\frac{1}{2m}\left(i\frac{\partial}{\partial t} - m + m\Phi_G\right)\left(\frac{1+2\Phi_G}{2m}\right)\hat{\mathbf{p}}\right]\phi + m\phi + m\Phi_G\phi \\
&- i\boldsymbol{\tau}\cdot\left[(1+2\Phi_G)\hat{\mathbf{p}}\times\frac{1}{2m}\left(i\frac{\partial}{\partial t} - m + m\Phi_G\right)\left(\frac{1+2\Phi_G}{2m}\right)\hat{\mathbf{p}}\right]\phi
\end{aligned}$$

However, in the above equation,

$$(1+2\Phi_G)\hat{\mathbf{p}}\times\left(\frac{1+2\Phi_G}{2m}\right)\hat{\mathbf{p}} = (1+2\Phi_G)\hat{\mathbf{p}}\times\frac{1}{2m}\left(i\frac{\partial}{\partial t} - m + m\Phi_G\right)\left(\frac{1+2\Phi_G}{2m}\right)\hat{\mathbf{p}} = 0$$

Since, both the expressions involve cross product of terms that is proportional to $\hat{\mathbf{p}}$, and the cross product of a vector with itself is zero. Thus, the equation for $i\frac{\partial\phi}{\partial t}$ reduces to,

$$\begin{aligned}
i\frac{\partial\phi}{\partial t} =& \left[(1+2\Phi_G)\hat{\mathbf{p}}\cdot\left(\frac{1+2\Phi_G}{2m}\right)\hat{\mathbf{p}}\right]\phi + m\phi + m\Phi_G\phi \\
&- \left[(1+2\Phi_G)\hat{\mathbf{p}}\cdot\frac{1}{2m}\left(i\frac{\partial}{\partial t} - m + m\Phi_G\right)\left(\frac{1+2\Phi_G}{2m}\right)\hat{\mathbf{p}}\right]\phi
\end{aligned}$$

Now, $\Phi_G(z)\hat{\mathbf{p}}$ can be written as,

$$\Phi_G(z)\hat{\mathbf{p}} = \hat{\mathbf{p}}\Phi_G(z) + [\Phi_G(z), \hat{\mathbf{p}}]$$

Let us try to evaluate the the commutator $[\Phi_G(z), \hat{\mathbf{p}}]$, using a test function ψ,

$$\begin{aligned}
[\Phi_G(z), \hat{\mathbf{p}}]\psi &= \Phi_G\hat{\mathbf{p}}\psi - \hat{\mathbf{p}}(\Phi_G\psi) \\
&= \Phi_G\hat{\mathbf{p}}\psi - \Phi_G\hat{\mathbf{p}}\psi - \hat{\mathbf{p}}(\Phi_G)\psi \\
&= -\hat{\mathbf{p}}(\Phi_G)\psi \\
\implies [\Phi_G(z), \hat{\mathbf{p}}] &= -\hat{\mathbf{p}}(\Phi_G) = i\boldsymbol{\nabla}\Phi_G
\end{aligned}$$

So, we can now write $\Phi_G(z)\hat{\mathbf{p}}$ as,

$$\Phi_G(z)\hat{\mathbf{p}} = \hat{\mathbf{p}}\Phi_G(z) + i\boldsymbol{\nabla}\Phi_G$$

Using the above relation, the equation for $i\frac{\partial\phi}{\partial t}$ can be modified as,

$$\begin{aligned}
i\frac{\partial\phi}{\partial t} =& \left[(1+2\Phi_G)\hat{\mathbf{p}}\cdot\left(\frac{1+2\Phi_G}{2m}\right)\hat{\mathbf{p}}\right]\phi + m\phi + m\Phi_G\phi \\
&- \left[(1+2\Phi_G)\hat{\mathbf{p}}\cdot\frac{1}{2m}\left(i\frac{\partial}{\partial t} - m + m\Phi_G\right)\left(\frac{1+2\Phi_G}{2m}\right)\hat{\mathbf{p}}\right]\phi \\
=& \left[\frac{\hat{\mathbf{p}}}{2m}\cdot(1+2\Phi_G)^2\hat{\mathbf{p}} + \frac{i}{2m}\boldsymbol{\nabla}\Phi_G\cdot(1+2\Phi_G)\hat{\mathbf{p}}\right]\phi + m\phi + m\Phi_G\phi \\
&- \left\{(1+2\Phi_G)\hat{\mathbf{p}}\cdot\frac{1}{2m}\left(\frac{1+2\Phi_G}{2m}\right)\left[\hat{\mathbf{p}}\left(i\frac{\partial}{\partial t} - m + m\Phi_G\right) + im\boldsymbol{\nabla}\Phi_G\right]\right\}\phi \\
=& \left[\frac{\hat{\mathbf{p}}}{2m}\cdot(1+4\Phi_G)\hat{\mathbf{p}}\right]\phi + m\phi + m\Phi_G\phi + \left[\frac{i}{m}\boldsymbol{\nabla}\Phi_G\cdot(1+2\Phi_G)\hat{\mathbf{p}}\right]\phi \\
&- \left[(1+2\Phi_G)\hat{\mathbf{p}}\cdot\frac{1}{2m}\left(\frac{1+2\Phi_G}{2m}\right)\hat{\mathbf{p}}\left(i\frac{\partial}{\partial t} - m + m\Phi_G\right)\right]\phi \\
&- \left[(1+2\Phi_G)\hat{\mathbf{p}}\cdot\frac{1}{4m}(1+2\Phi_G)i\boldsymbol{\nabla}\Phi_G\right]\phi
\end{aligned}$$

Let us consider the term,

$$\left(i\frac{\partial}{\partial t} - m + m\Phi_G\right)\phi$$

Now, the partial differentiation with respect to time represents the operator for the total energy of the system. This accounts for the kinetic energy, potential energy and the mass-energy of the system. Since there is already a negative of the mass-energy in the above expression, it can be re-written as,

$$\left(i\frac{\partial}{\partial t} - m + m\Phi_G\right)\phi = \left(\frac{\hat{p}^2}{2m} + 2m\Phi_G\right)\phi$$

Thus, the equation concerning $i\frac{\partial\phi}{\partial t}$ becomes,

$$\begin{aligned}
i\frac{\partial\phi}{\partial t} =& \left[\frac{\hat{\mathbf{p}}}{2m}\cdot(1+4\Phi_G)\hat{\mathbf{p}}\right]\phi + m\phi + m\Phi_G\phi + \left[\frac{i}{m}\boldsymbol{\nabla}\Phi_G\cdot(1+2\Phi_G)\hat{\mathbf{p}}\right]\phi \\
&- \left[(1+2\Phi_G)\hat{\mathbf{p}}\cdot\frac{1}{2m}\left(\frac{1+2\Phi_G}{2m}\right)\hat{\mathbf{p}}\left(\frac{\hat{p}^2}{2m}+2m\Phi_G\right)\right]\phi \\
&- \left[(1+2\Phi_G)\hat{\mathbf{p}}\cdot\frac{1}{4m}(1+2\Phi_G)i\boldsymbol{\nabla}\Phi_G\right]\phi \\
=& \left[\frac{\hat{\mathbf{p}}}{2m}\cdot(1+4\Phi_G)\hat{\mathbf{p}}\right]\phi + m\phi + m\Phi_G\phi + \frac{i}{m}(\boldsymbol{\nabla}\Phi_G\cdot\hat{\mathbf{p}})\phi \\
&- \left[\hat{\mathbf{p}}\cdot\left(\frac{1+4\Phi_G}{4m^2}\right)\hat{\mathbf{p}} + i\boldsymbol{\nabla}\Phi_G\cdot\left(\frac{1+2\Phi_G}{2m^2}\right)\hat{\mathbf{p}}\right]\left[\frac{\hat{p}^2}{2m}+2m\Phi_G\right]\phi \\
&- \left[\frac{i}{4m}\boldsymbol{\nabla}\Phi_G\cdot(1+4\Phi_G)\hat{\mathbf{p}}\right]\phi \\
=& \left[\frac{\hat{\mathbf{p}}}{2m}\cdot(1+4\Phi_G)\hat{\mathbf{p}}\right]\phi + m\phi + m\Phi_G\phi + \frac{i}{m}(\boldsymbol{\nabla}\Phi_G\cdot\hat{\mathbf{p}})\phi \\
&- \left[\hat{\mathbf{p}}\cdot\left(\frac{1+4\Phi_G}{4m^2}\right)\hat{\mathbf{p}}\right]\left[\frac{\hat{p}^2}{2m}+2m\Phi_G\right]\phi \\
&- \left[i\boldsymbol{\nabla}\Phi_G\cdot\left(\frac{1+2\Phi_G}{2m^2}\right)\hat{\mathbf{p}}\right]\left[\frac{\hat{p}^2}{2m}\right]\phi - \left[\frac{i}{4m}\boldsymbol{\nabla}\Phi_G\cdot\hat{\mathbf{p}}\right]\phi \\
=& \left[\frac{\hat{\mathbf{p}}}{2m}\cdot(1+4\Phi_G)\hat{\mathbf{p}}\right]\phi + m\phi + m\Phi_G\phi + \frac{3i}{4m}(\boldsymbol{\nabla}\Phi_G\cdot\hat{\mathbf{p}})\phi \\
&- \left[\hat{\mathbf{p}}\cdot\left(\frac{1+4\Phi_G}{4m^2}\right)\left(\frac{\hat{p}^2}{2m}+2m\Phi_G\right)\hat{\mathbf{p}} - \hat{\mathbf{p}}\cdot\left(\frac{1+4\Phi_G}{4m^2}\right)2mi\boldsymbol{\nabla}\Phi_G\right]\phi \\
&- \left[i\boldsymbol{\nabla}\Phi_G\cdot\hat{\mathbf{p}}\right]\left[\frac{\hat{p}^2}{4m^3}\right]\phi \\
=& \left[\frac{\hat{\mathbf{p}}}{2m}\cdot(1+4\Phi_G)\hat{\mathbf{p}}\right]\phi + m\phi + m\Phi_G\phi + \frac{3i}{4m}(\boldsymbol{\nabla}\Phi_G\cdot\hat{\mathbf{p}})\phi \\
&- \left[\hat{\mathbf{p}}\cdot\left(\frac{1+4\Phi_G}{4m^2}\right)\left(\frac{\hat{p}^2}{2m}+2m\Phi_G\right)\hat{\mathbf{p}}\right]\phi + \left[\hat{\mathbf{p}}\cdot\left(\frac{1}{2m}\right)i\boldsymbol{\nabla}\Phi_G\right]\phi - \left[i\boldsymbol{\nabla}\Phi_G\cdot\hat{\mathbf{p}}\right]\left[\frac{\hat{p}^2}{4m^3}\right]\phi \\
=& \left[\frac{\hat{\mathbf{p}}}{2m}\cdot(1+4\Phi_G)\hat{\mathbf{p}}\right]\phi + m\phi + m\Phi_G\phi + \frac{5i}{4m}(\boldsymbol{\nabla}\Phi_G\cdot\hat{\mathbf{p}})\phi \\
&- \left[\hat{\mathbf{p}}\cdot\left(\frac{\Phi_G}{2m}\right)\hat{\mathbf{p}} + \frac{\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}}{8m^3}\hat{p}^2 + \hat{\mathbf{p}}\cdot\left(\frac{\Phi_G}{2m^3}\right)\hat{p}^2\hat{\mathbf{p}}\right]\phi - \left[i\boldsymbol{\nabla}\Phi_G\cdot\hat{\mathbf{p}}\right]\left[\frac{\hat{p}^2}{4m^3}\right]\phi \\
=& \left[\frac{\hat{\mathbf{p}}}{2m}\cdot(1+3\Phi_G)\hat{\mathbf{p}}\right]\phi + m\phi + m\Phi_G\phi - \frac{\hat{p}^4}{8m^3} + \frac{5i}{4m}(\boldsymbol{\nabla}\Phi_G\cdot\hat{\mathbf{p}})\phi \\
&- \left[(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}})\Phi_G + i\boldsymbol{\nabla}\Phi_G\cdot\hat{\mathbf{p}}\right]\left(\frac{\hat{p}^2}{2m^3}\right)\phi - \left[i\boldsymbol{\nabla}\Phi_G\cdot\hat{\mathbf{p}}\right]\left[\frac{\hat{p}^2}{4m^3}\right]\phi \\
\therefore i\frac{\partial\phi}{\partial t} =& \left[\frac{\hat{\mathbf{p}}}{2m}\cdot(1+3\Phi_G)\hat{\mathbf{p}}\right]\phi + m\phi + m\Phi_G\phi - \frac{\hat{p}^4}{8m^3}\phi + \frac{5i}{4m}(\boldsymbol{\nabla}\Phi_G\cdot\hat{\mathbf{p}})\phi \\
&- \left[\left(\hat{p}^2\right)\Phi_G\right]\left[\frac{\hat{p}^2}{2m^3}\right]\phi - \left[i\boldsymbol{\nabla}\Phi_G\cdot\hat{\mathbf{p}}\right]\left[\frac{3\hat{p}^2}{4m^3}\right]\phi
\end{aligned}$$

Now, let us consider the term $\left[(\hat{p}^2)\Phi_G\right]\left[\frac{\hat{p}^2}{2m^3}\right]$,

$$\begin{aligned}
\left[(\hat{p}^2)\Phi_G\right]\left[\frac{\hat{p}^2}{2m^3}\right] &= \frac{\hat{p}^2}{2m^3}\left[\Phi_G(\hat{p}^2)\right] \\
&= \frac{\hat{p}^2}{2m^3}\left[\Phi_G(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}})\right] \\
&= \frac{\hat{p}^2}{2m^3}\left[\hat{\mathbf{p}}\cdot\Phi_G\hat{\mathbf{p}} + i\boldsymbol{\nabla}\Phi_G\cdot\hat{\mathbf{p}}\right] \\
&= \frac{\hat{p}^2}{2m^3}\left[\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}\Phi_G + \hat{\mathbf{p}}\cdot i\boldsymbol{\nabla}\Phi_G + i\boldsymbol{\nabla}\Phi_G\cdot\hat{\mathbf{p}}\right] \\
\therefore \left[(\hat{p}^2)\Phi_G\right]\left[\frac{\hat{p}^2}{2m^3}\right] &= \frac{\hat{p}^2}{2m^3}\left[\hat{p}^2\Phi_G + 2i\boldsymbol{\nabla}\Phi_G\cdot\hat{\mathbf{p}}\right]
\end{aligned}$$

And hence, the equation for $i\frac{\partial\phi}{\partial t}$ further modifies to,

$$\begin{aligned}
i\frac{\partial\phi}{\partial t} =& \left[\frac{\hat{\mathbf{p}}}{2m}\cdot(1+3\Phi_G)\hat{\mathbf{p}}\right]\phi + m\phi + m\Phi_G\phi - \frac{\hat{p}^4}{8m^3}\phi - \frac{\hat{p}^4}{2m^3}\Phi_G\phi \\
&+ \left(\frac{5}{4} - \frac{\hat{p}^2}{m^2}\right)\left(\frac{i\boldsymbol{\nabla}\Phi_G}{m}\cdot\hat{\mathbf{p}}\right)\phi - \left(\frac{i\boldsymbol{\nabla}\Phi_G}{m}\cdot\hat{\mathbf{p}}\right)\left(\frac{3\hat{p}^2}{4m^2}\right)\phi
\end{aligned}$$

The non-Hermitian terms that are proportional to $i\boldsymbol{\nabla}\Phi_G\cdot\hat{\mathbf{p}}$ arise due to the fact that the density of χ varies as the particle accelerates. These non-Hermitian terms can be removed by using the following transformation of the wave function,

$$\phi \to \phi\left(1 + \frac{5\hat{p}^2}{8m^2}\right)$$

Furthermore, ignoring the higher order term proportional to $\hat{p}^4\Phi_G\phi$, we obtain the expression that looks similar to the non-relativistic Schrödinger equation,

$$i\frac{\partial\phi}{\partial t} = \left[\frac{\hat{\mathbf{p}}}{2m}\cdot(1+3\Phi_G)\hat{\mathbf{p}}\right]\phi + m(1+\Phi_G)\phi - \frac{\hat{p}^4}{8m^3}\phi \tag{5.39}$$

In the Heisenberg picture, the rate of change of position can be defined as,

$$\hat{v} \equiv \frac{\mathrm{d}\hat{z}}{\mathrm{d}t} = i\left[\hat{H},\hat{z}\right] = \frac{1}{i}\left[\hat{z},\hat{H}\right] \tag{5.40}$$

Here, $\hat{z}$ is the position operator along the z-axis and $\hat{H}$ represents the Hamiltonian of the particle. The Hamiltonian of the electron can be found in equation (5.39), since it represents a Schrödinger-like equation,

$$\hat{H} = \frac{1}{2m}\hat{\mathbf{p}}\cdot[(1+3\Phi_G)\hat{\mathbf{p}}] + (1+\Phi_G)m - \frac{1}{8m^3}\hat{p}^4 \tag{5.41}$$

Substituting equation (5.41) into equation (5.40), we get,

$$
\begin{aligned}
\hat{v} \equiv \frac{\mathrm{d}\hat{z}}{\mathrm{d}t} =& \frac{1}{i}\left[\hat{z}, \left(\frac{1}{2m}\hat{\mathbf{p}} \cdot [(1+3\Phi_G)\hat{\mathbf{p}}] + (1+\Phi_G)m - \frac{1}{8m^3}\hat{p}^4\right)\right] \\
=& \frac{1}{i}\left[\hat{z}, \frac{1}{2m}\hat{\mathbf{p}} \cdot [(1+3\Phi_G)\hat{\mathbf{p}}]\right] + \frac{1}{i}[\hat{z}, (1+\Phi_G)m] - \frac{1}{i}\left[\hat{z}, \frac{\hat{p}^4}{8m^3}\right] \\
=& \frac{1}{2im}[\hat{z}, \hat{\mathbf{p}} \cdot [(1+3\Phi_G)\hat{\mathbf{p}}]] + \frac{1}{i}[z(1+\Phi_G)m - (1+\Phi_G)zm] - \frac{1}{i}\left[\hat{z}, \frac{\hat{p}^4}{8m^3}\right] \\
=& \frac{1}{2im}\left\{[\hat{z}, \hat{\mathbf{p}}] \cdot (1+3\Phi_G)\hat{\mathbf{p}} + \hat{\mathbf{p}} \cdot [\hat{z}, (1+3\Phi_G)\hat{\mathbf{p}}]\right\} - \frac{1}{8im^3}\left\{[\hat{z}, \hat{p}^2]\hat{p}^2 + \hat{p}^2[\hat{z}, \hat{p}^2]\right\} \\
=& \frac{1}{2im}\left\{[\hat{z}, \hat{\mathbf{p}}] \cdot (1+3\Phi_G)\hat{\mathbf{p}} + \hat{\mathbf{p}} \cdot (1+3\Phi_G)[\hat{z}, \hat{\mathbf{p}}]\right\} \\
&- \frac{1}{8im^3}\left\{\big[[\hat{z}, \hat{\mathbf{p}}] \cdot \hat{\mathbf{p}}\big]\hat{p}^2 + \big[\hat{\mathbf{p}} \cdot [\hat{z}, \hat{\mathbf{p}}]\big]\hat{p}^2 + \hat{p}^2\big[[\hat{z}, \hat{\mathbf{p}}] \cdot \hat{\mathbf{p}}\big] + \hat{p}^2\big[\hat{\mathbf{p}} \cdot [\hat{z}, \hat{\mathbf{p}}]\big]\right\}
\end{aligned}
$$

Now,

$$[\hat{z}, \hat{p}_x] = [\hat{z}, \hat{p}_y] = 0 \quad [\hat{z}, \hat{p}_z] = i$$

while considering $\hbar = 1$. Thus, the equation for $\hat{v}$ becomes,

$$
\begin{aligned}
\hat{v} \equiv \frac{\mathrm{d}\hat{z}}{\mathrm{d}t} =& \frac{1}{2im}\left\{[\hat{z}, \hat{p}_z](1+3\Phi_G)\hat{p}_z + \hat{p}_z(1+3\Phi_G)[\hat{z}, \hat{p}_z]\right\} \\
&- \frac{1}{8im^3}\left\{\big[[\hat{z}, \hat{p}_z]\hat{p}_z\big]\hat{p}^2 + \big[\hat{p}_z[\hat{z}, \hat{p}_z]\big]\hat{p}^2 + \hat{p}^2\big[[\hat{z}, \hat{p}_z]\hat{p}_z\big] + \hat{p}^2\big[\hat{p}_z[\hat{z}, \hat{p}_z]\big]\right\} \\
=& \frac{1}{2m}\big[\hat{p}_z(1+3\Phi_G) + (1+3\Phi_G)\hat{p}_z\big] - \frac{1}{8m^3}\big[2\hat{p}_z\hat{p}^2 + 2\hat{p}^2\hat{p}_z\big]
\end{aligned}
$$

Now,

$$\hat{p}_z\hat{p}^2 = \left(\frac{1}{i}\frac{\partial}{\partial z}\right)\left(\frac{\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}}{i^2}\right) = \left(\frac{\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}}{i^2}\right)\left(\frac{1}{i}\frac{\partial}{\partial z}\right) = \hat{p}^2\hat{p}_z$$

Thus, the equation for $\hat{v}$ becomes,

$$
\begin{aligned}
\hat{v} =& \frac{1}{2m}\big[\hat{p}_z(1+3\Phi_G) + (1+3\Phi_G)\hat{p}_z\big] - \frac{1}{8m^3}\big[4\hat{p}^2\hat{p}_z\big] \\
=& \frac{1}{2m}\big[\hat{p}_z(1+3\Phi_G) + (1+3\Phi_G)\hat{p}_z\big] - \frac{1}{2m^3}\big[\hat{p}^2\hat{p}_z\big] \\
\therefore \hat{v} =& \frac{1}{2m}\big[\hat{p}_z(1+3\Phi_G) + (1+3\Phi_G)\hat{p}_z\big] - \frac{1}{2}\left(\frac{\hat{p}^2}{m^2}\right)\left(\frac{\hat{p}_z}{m}\right)
\end{aligned}
\tag{5.42}
$$

Now, the acceleration of the particle can also be written as,

$$\hat{a} \equiv \frac{\mathrm{d}\hat{v}}{\mathrm{d}t} = i\left[\hat{H}, \hat{v}\right] = \frac{1}{i}\left[\hat{v}, \hat{H}\right] \tag{5.43}$$

Substituting equation (5.41) and equation (5.42) into equation (5.43), we have,

$$\begin{aligned}
\hat{a} \equiv \frac{\mathrm{d}\hat{v}}{\mathrm{d}t} =& \frac{1}{i}\Bigg[\frac{1}{2m}\left[\hat{p}_z(1+3\Phi_G)+(1+3\Phi_G)\hat{p}_z\right]-\frac{1}{2}\left(\frac{\hat{p}^2}{m^2}\right)\left(\frac{\hat{p}_z}{m}\right) \\
&, \frac{1}{2m}\hat{\mathbf{p}}\cdot[(1+3\Phi_G)\hat{\mathbf{p}}]+(1+\Phi_G)m-\frac{1}{8m^3}\hat{p}^4\Bigg] \\
=& \frac{1}{i}\Bigg[\frac{1}{4m^2}\Big[\hat{p}_z(1+3\Phi_G),\hat{\mathbf{p}}\cdot(1+3\Phi_G)\hat{\mathbf{p}}\Big]+\frac{1}{4m^2}\Big[(1+3\Phi_G)\hat{p}_z,\hat{\mathbf{p}}\cdot(1+3\Phi_G)\hat{\mathbf{p}}\Big] \\
&+\frac{1}{2}\Big[\hat{p}_z(1+3\Phi_G),(1+\Phi_G)\Big]+\frac{1}{2}\Big[(1+3\Phi_G)\hat{p}_z,(1+\Phi_G)\Big] \\
&-\frac{1}{16m^4}\Big[\hat{p}_z(1+3\Phi_G),\hat{p}^4\Big]-\frac{1}{16m^4}\Big[(1+3\Phi_G)\hat{p}_z,\hat{p}^4\Big] \\
&-\frac{1}{4m^4}\Big[(\hat{p})^2\hat{p}_z,\hat{\mathbf{p}}\cdot(1+3\Phi_G)\hat{\mathbf{p}}\Big]-\frac{1}{2m^2}\Big[(\hat{p})^2\hat{p}_z,(1+\Phi_G)\Big]+\frac{1}{16m^6}\Big[(\hat{p})^2\hat{p}_z,\hat{p}^4\Big]\Bigg] \\
=& \frac{1}{i}\Bigg[\frac{1}{4m^2}\Big[\hat{p}_z(1+3\Phi_G),\hat{\mathbf{p}}\cdot(1+3\Phi_G)\hat{\mathbf{p}}\Big]+\frac{1}{4m^2}\Big[(1+3\Phi_G)\hat{p}_z,\hat{\mathbf{p}}\cdot(1+3\Phi_G)\hat{\mathbf{p}}\Big] \\
&+\frac{1}{2}\Big[\hat{p}_z,\Phi_G\Big]+\frac{1}{2}\Big[\hat{p}_z,\Phi_G\Big]-\frac{1}{16m^4}\Big[\hat{p}_z(1+3\Phi_G),\hat{p}^4\Big]-\frac{1}{16m^4}\Big[(1+3\Phi_G)\hat{p}_z,\hat{p}^4\Big] \\
&-\frac{1}{4m^4}\Big[(\hat{p}^2)\hat{p}_z,\hat{\mathbf{p}}\cdot(1+3\Phi_G)\hat{\mathbf{p}}\Big]-\frac{1}{2m^2}\Big[(\hat{p})^2\hat{p}_z,\Phi_G\Big]+\frac{1}{16m^6}\Big[(\hat{p})^2\hat{p}_z,\hat{p}^4\Big]\Bigg]
\end{aligned}$$

while retaining only those terms that are of the first order in Φ_G. Since $[\hat{p}_z,\hat{\mathbf{p}}]=0$, then,

$$\begin{aligned}
\Big[(\hat{p})^2\hat{p}_z,\hat{p}^4\Big] =& (\hat{p})^2\left[\hat{p}_z,\hat{p}^2\hat{p}^2\right]+\left[\hat{p}^2,\hat{p}^2\hat{p}^2\right]\hat{p}_z \\
=& (\hat{p})^4\left[\hat{p}_z,\hat{p}^2\right]+(\hat{p})^2\left[\hat{p}_z,\hat{p}^2\right](\hat{p})^2+\hat{p}^2\left[\hat{p}^2,\hat{p}^2\right]\hat{p}_z+\left[\hat{p}^2,\hat{p}^2\right]\hat{p}^2\hat{p}_z \\
=& (\hat{p})^4[\hat{p}_z,\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}]+(\hat{p})^2[\hat{p}_z,\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}](\hat{p})^2+0 \\
=& (\hat{p})^4\hat{\mathbf{p}}\cdot[\hat{p}_z,\hat{\mathbf{p}}]+(\hat{p})^4[\hat{p}_z,\hat{\mathbf{p}}]\cdot\hat{\mathbf{p}}+(\hat{p})^2\hat{\mathbf{p}}\cdot[\hat{p}_z,\hat{\mathbf{p}}](\hat{p})^2+(\hat{p})^2[\hat{p}_z,\hat{\mathbf{p}}]\cdot\hat{\mathbf{p}}(\hat{p})^2 \\
\therefore\ \Big[(\hat{p})^2\hat{p}_z,\hat{p}^4\Big] =& 0
\end{aligned}$$

Evaluating the other commutators present in the acceleration equation,

$$\begin{aligned}
[\hat{p}_z(1+3\Phi_G),\hat{\mathbf{p}}\cdot(1+3\Phi_G)\hat{\mathbf{p}}] =& [\hat{p}_z,\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}]+3[\hat{p}_z\Phi_G,\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}]+3[\hat{p}_z,\hat{\mathbf{p}}\cdot\Phi_G\hat{\mathbf{p}}] \\
=& 0+3\Big[\hat{\mathbf{p}}\cdot[\hat{p}_z\Phi_G,\hat{\mathbf{p}}]+[\hat{p}_z\Phi_G,\hat{\mathbf{p}}]\cdot\hat{\mathbf{p}}\Big]+3\Big[\hat{\mathbf{p}}\cdot[\hat{p}_z,\Phi_G\hat{\mathbf{p}}]+0\Big] \\
=& 3\Big[\hat{\mathbf{p}}\cdot\hat{p}_z[\Phi_G,\hat{\mathbf{p}}]+\hat{p}_z[\Phi_G,\hat{\mathbf{p}}]\cdot\hat{\mathbf{p}}+\hat{\mathbf{p}}\cdot[\hat{p}_z,\Phi_G]\hat{\mathbf{p}}\Big] \\
=& 3\Big[\hat{\mathbf{p}}\cdot\hat{p}_z i\boldsymbol{\nabla}\Phi_G+\hat{p}_z i\boldsymbol{\nabla}\Phi_G\cdot\hat{\mathbf{p}}-i\nabla\Phi_G\hat{p}^2\Big] \\
\therefore\ [\hat{p}_z(1+3\Phi_G),\hat{\mathbf{p}}\cdot(1+3\Phi_G)\hat{\mathbf{p}}] =& 6\Big[\hat{p}_z i\boldsymbol{\nabla}\Phi_G\cdot\hat{\mathbf{p}}\Big]-3i\nabla\Phi_G\hat{p}^2
\end{aligned}$$

$$\begin{aligned}
[(1+3\Phi_G)\hat{p}_z,\hat{\mathbf{p}}\cdot(1+3\Phi_G)\hat{\mathbf{p}}] =& [\hat{p}_z,\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}]+3[\Phi_G\hat{p}_z,\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}]+3[\hat{p}_z,\hat{\mathbf{p}}\cdot\Phi_G\hat{\mathbf{p}}] \\
=& 0+3\Big[0+[\Phi_G,\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}]\hat{p}_z\Big]+3\Big[\hat{\mathbf{p}}\cdot[\hat{p}_z,\Phi_G\hat{\mathbf{p}}]+0\Big] \\
=& 6i\boldsymbol{\nabla}\Phi_G\cdot\hat{\mathbf{p}}\hat{p}_z+3\hat{\mathbf{p}}\cdot[\hat{p}_z,\Phi_G]\hat{\mathbf{p}} \\
\therefore\ [(1+3\Phi_G)\hat{p}_z,\hat{\mathbf{p}}\cdot(1+3\Phi_G)\hat{\mathbf{p}}] =& 6\Big[\hat{p}_z i\boldsymbol{\nabla}\Phi_G\cdot\hat{\mathbf{p}}\Big]-3i\nabla\Phi_G\hat{p}^2
\end{aligned}$$

$$\begin{aligned}
\left[(\hat{p})^2\hat{p}_z, \Phi_G\right] =& (\hat{p})^2[\hat{p}_z, \Phi_G] + \left[(\hat{p})^2, \Phi_G\right]\hat{p}_z \\
=& -(\hat{p})^2[i\nabla\Phi_G] - 2i\boldsymbol{\nabla}\Phi_G \cdot \hat{\mathbf{p}}\hat{p}_z \\
\therefore \left[(\hat{p})^2\hat{p}_z, \Phi_G\right] =& -2\hat{p}_z i\boldsymbol{\nabla}\Phi_G \cdot \hat{\mathbf{p}} - i\nabla\Phi_G\hat{p}^2
\end{aligned}$$

Substituting the above values and considering only the first-order terms in the acceleration $\hat{a}$ equation, we get,

$$\begin{aligned}
\hat{a} =& \frac{1}{i}\left[\frac{1}{4m^2}\left[6\hat{p}_z i\boldsymbol{\nabla}\Phi_G \cdot \hat{\mathbf{p}} - 3i\nabla\Phi_G\hat{p}^2\right] + \frac{1}{4m^2}\left[6\hat{p}_z i\boldsymbol{\nabla}\Phi_G \cdot \hat{\mathbf{p}} - 3i\nabla\Phi_G\hat{p}^2\right]\right. \\
&\left. + [\hat{p}_z, \Phi_G] - \frac{1}{2m^2}\left[-2\hat{p}_z i\boldsymbol{\nabla}\Phi_G \cdot \hat{\mathbf{p}} - i\nabla\Phi_G\hat{p}^2\right]\right] \\
=& \left[\frac{1}{2m^2}\left[8\hat{p}_z\boldsymbol{\nabla}\Phi_G \cdot \hat{\mathbf{p}} - 2\nabla\Phi_G\hat{p}^2\right] - \nabla\Phi_G\right] \\
=& \left[-\nabla\Phi_G + \frac{1}{m^2}\left(4\hat{p}_z\boldsymbol{\nabla}\Phi_G \cdot \hat{\mathbf{p}} - \nabla\Phi_G\hat{p}^2\right)\right] \\
=& \left[-\nabla\Phi_G + \frac{1}{m^2}\left(4\hat{p}_z\boldsymbol{\nabla}\Phi_G \cdot \hat{\mathbf{p}} - \nabla\Phi_G\hat{p}^2\right)\right] \\
=& \left[-\nabla\Phi_G + \frac{1}{m^2}\left(4\hat{p}_z\nabla\Phi_G\hat{p}_z - \nabla\Phi_G\hat{p}^2\right)\right] \\
=& -\nabla\Phi_G\left[1 + \frac{1}{m^2}\left[-4(\hat{p}_z)^2 + (\hat{p}_x)^2 + (\hat{p}_y)^2 + (\hat{p}_z)^2\right]\right] \\
\therefore \hat{a} =& -\nabla\Phi_G\left[1 + \frac{1}{m^2}\left[(\hat{p}_x)^2 + (\hat{p}_y)^2 - 3(\hat{p}_z)^2\right]\right]
\end{aligned}$$

Taking the expectation value of the above equation, we get,

$$a = -\nabla\Phi_G\left[1 + \frac{1}{m^2}\left({p_x}^2 + {p_y}^2 - 3{p_z}^2\right)\right] \tag{5.44}$$

where the notations ${p_x}^2$, ${p_y}^2$, etc have been written instead of the usual notations for expectation value $\langle {p_x}^2 \rangle$, $\langle {p_y}^2 \rangle$, etc, for simplicity. These quantities correspond to the observables in their respective classical sense. Let us now re-write the following quantities,

$$\frac{{p_x}^2}{m^2} = {v_x}^2, \quad \frac{{p_y}^2}{m^2} = {v_y}^2, \quad \frac{{p_z}^2}{m^2} = {v_z}^2$$

Now, equation (5.44) becomes,

$$a = -\nabla\Phi_G\left[1 + \left({v_x}^2 + {v_y}^2 - 3{v_z}^2\right)\right] \tag{5.45}$$

But since we have considered the electron to be moving along the z-axis direction only, we have,

$${v_x}^2 = {v_y}^2 = 0, \quad {v_z}^2 = v^2$$

Thus, equation (5.45) further modifies to,

$$a = -\nabla\Phi_G\left[1 - 3v^2\right] \tag{5.46}$$

The equation (5.46) represents the non-relativistic acceleration of the electron in a spherically symmetric and static spacetime. Since we had considered natural units in which $c = 1$, if we took the SI system, then the equation for acceleration would become,

$$a = -\nabla\Phi_G \left[1 - 3\left(\frac{v^2}{c^2}\right)\right] \tag{5.47}$$

In comparison, the acceleration equation that would be obtained from the Newtonian gravity, as shown in equation (3.33) would be,

$$a = -\nabla\Phi_G \tag{5.48}$$

The results obtained from equation (5.47) and from equation (5.48) has been compared in the following plot,

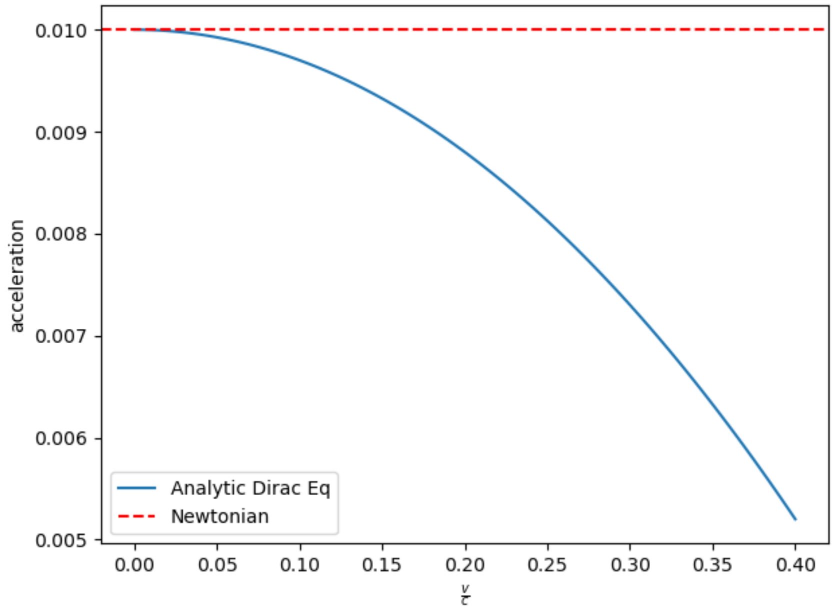

Figure 5.1: Plot representing the comparison between the acceleration obtained from the analytical calculation of the Dirac equation and from the Newtonian gravity, for $\nabla\Phi_G = -1$.

Chapter 6

Literature Review

In this project report, we have briefly described the development of the Dirac equation, including a detail of the Lie group and the representation of Lorentz group and its application in calculating the acceleration of the electron in a Schwarzschild spacetime. Whilst there is no generally accepted way of effectively understanding the Quantum Field theory, I have used the textbook, 'An Introduction to Quantum Field theory' by Michael E. Peskin and Daniel E. Schroeder [1], which is used by a large majority of Quantum field theorists. The textbook advocates a good deal of physical intutition, along with the necessary mathematical rigour. A more comprehensive mathematical approach was done in Tobias J. Osborne's 'Lectures on Quantum Field theory' [2], which was done during the winter semester of the year, 2016-17. Additional reading involved the use of the famous Dirac's paper, 'The Quantum Theory of Electron' [3], which introduced the Dirac equation as well as use of the textbooks, 'Quantum Field Theory' by Lowell S. Brown [5], 'Quantum Field Theory for Gifted Amateurs' by Tom Lancaster and Stephen J. Blundell [6], and 'Field Theory: A Path integral approach' by Ashok Das [8].

In order to determine the acceleration of the electron in curved spacetime, much of the guidance was provided by the paper 'Numerical calculation of the relativistic acceleration of an electronin curved spacetime using the Dirac equation' by J. D. Franson [7]. The study of a number of topics were done using various resources, like 'Gravitation and Cosmology' by Steven Weinberg [11] for Lorentz group representations and Tetrad formalism, 'Gravitation and Spacetime' by Hans C. Ohanian [13] for spacetime geometry and the Schwarzschild solution of the Einstein field equation, 'Lectures on Quantum Mechanics' by Gordon Baym [14] for the transformation properties of spin. Additional reading was also done from 'An Introduction to Relativistic Quantum Field Theory' by S. S. Schweber [12] and the paper titled 'General Relativistic Effects of Gravity in Quantum Mechanics' by K. Konno and M. Kasai [9]. Reading on the top of spin connection was also done from the papaer titled 'Note about the Spin Connection in General Relativity' by Renata Jora [10].

A review of the mathematical processes and methods that were used for this project, including contour integration and differential vector operators, was done using 'Mathematical Methods for Physicists: A Comprehensive Guide' by Arfken, Weber and Harris [4].

Chapter 7

Conclusion

Quantum Field Theory is a result of decades long scientific research by a number of theoretical physicists, which began with Dirac's attempt to quantize the electromagnetic field, towards the late 1920s. Although the roots of the formulation of this theory began with Oskar Klein and Walter Gordon bringing forth their Klein-Gordon equation, Paul Dirac, by introducing his breakthrough equation, effectively managed to combine elements of Quantum Mechanics and Special Relativity to bring out a relativistic Quantum Field Theory. This has been so successful that it had introduced the renormalized Quantum Electrodynamics. Further attempts were made to apply the same basic concepts to other fundamental forces, in which, all of them have been successful except for gravitational force.

With this project, I have made an attempt to apply the Dirac equation, to determine the motion of an electron in a gravitational field. Although the procedure does involve an approximation, the result shows a significant difference between the one obtained from Newtonian gravity and the one obtained using the Dirac equation in curved spacetime. The iterative procedure used in this case is a method that is commonly used in special relativity, especially for cases involving electric and magnetic fields, with the difference being that in this case, we have used gravitational field. The result of this analytical calculation is as per the paper titled "Numerical calculation of the relativistic acceleration of an electron in curved spacetime using the Dirac equation", by J. D. Franson[7], University of Maryland. It highlights the capacity of the Dirac equation to provide more accurate answers than the earlier classical methods.

It is also interesting to note that acceleration for the electron is proportional $\left(1 - 3v^2/c^2\right)$ in Schwarzschild spacetime, whereas in Minkowski spacetime, the acceleration of the particle is proportional $\left(1 - v^2/c^2\right)^{1/2}$ as per special relativity. This proportionality in the case of the Schwarzschild spacetime, is linked to the appearance of the quantity $(1 + 3\Phi_G)$ in the kinetic energy term of equation (5.39). It goes to show that for velocities higher than $c/\sqrt{3}$, the sign of the acceleration switches. This is a consequence of the fact that the time coordinate t is arbitrary and does not exactly correspond to the proper time τ.

Quantum Field theory has not only played a key role in elementary particle physics, but it has also found applications in many other fields like Condensed Matter Physics, Quantum Electrodynamics, etc, and has lead to the development of fields like Conformal Field Theory, String Theory, the Standard Model of particle physics, and so on. The development of Quantum Field Theory was therefore, crucial in opening up new boundaries in Physics, with current major developments being focused on String Theory and Loop Quantum Gravity.

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