PHASES OF THE HUBBARD AND SACHDEV YE KITAEV MODEL

Project Report

submitted in partial fulfillment of the requirement for the degree of

MASTER OF SCIENCE

IN

PHYSICS

by

ARYA ANTHERJANAM V

Reg No:16400216PH03

Under the guidance of

Dr.DEEPAK VAID

DEPARTMENT OF PHYSICS

NATIONAL INSTITUTE OF TECHNOLOGY

KARNATAKA,SURATHKAL,MANGALORE-575025

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DECLARATION

I hereby declare that the report of the P.G. Project Work entitled **"PHASES OF THE HUBBARD AND SACHDEV YE KITAEV MODEL"** which is submitted to National Institute of Technology Karnataka, Surathkal, in partial fulfillment of the requirements for the award of the Degree of Master of Science in the Department of Physics, is a bonafide report of the work carried out by me. The material contained in this report has not been submitted to any University or Institution for the award of any degree.

In keeping with the general practice in reporting scientific observations, due acknowledgement has been made whenever the work described is based on the findings of other investigators.

Place:
Date:

ARYA ANTHERJANAM V
Reg.No: 16400216PH03
Department of Physics
NITK,SURATHKAL

CERTIFICATE

This is to certify that the project entitled **"PHASES OF THE HUBBARD AND SACHDEV YE KITAEV MODEL"** is an authenticated record of work carried out by **ARYA ANTHERJANAM V** ,Reg.No:16400216PH03 in partial fulfillment of the requirement for the award of the Degree of Master of Science in Physics which is submitted to Department of Physics, National Institute of Technology, Karnataka, during the period 2017-2018.

Chairman-DPGC **Dr.DEEPAK VAID**

Project Supervisor

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Place:

Date:

ARYA ANTHERJANAM V

Reg.No: 16400216PH03

Department of Physics

NITK,SURATHKAL

ABSTRACT

Most of the theories on solids were built upon the assumption that the coulombic repulsion between electrons is negligible. But certain materials tend to show deviation from the predicted behaviour because of their strong correlation. Such materials demanded further study. Hubbard model is one such model which takes into account the lattice interactions between electrons. It was helpful in explaining about the Mott insulators, which were expected to be conductors, but showed an insulating behaviour. The model is written in terms of fermion creation and annihilation operators, similar to those discussed in quantum harmonic oscilllator. We study the Hamiltonian of the model in detail and make an attempt to describe a single site. Green's funciton for the model is studied ad analysed.

An object that crosses the event horizon of a black hole is fully absorbed and hence the information contained in it is lost forever. But according to quantum mechanics, information can neither be created nor be destroyed. This information paradox could be solved if we consider that all the information about the matter and energy within the three dimensional volume of a black hole is encoded as a hologram upon its two dimensional surface, the event horizon. This is termed as the holographic principle. Sachdev Ye Kitaev model emerged as a model for holography. It deals with random infinite-range interactions including Majorana fermions. The model is studied in detail and the Green's function for the model is determined and analysed.

Contents

List of Figures

Chapter 1

Introduction

The physical systems that we see everyday are ensembles of free particles. Most of the models used to explain the behaviour of solids asssume that the Coulomb energy of electrons is much smaller than their kinetic energy, and hence can be neglected. This suffices the description of numerous materials such as simple metals, semiconductors and insulators. But materials with open d- and f- electron shells, where electrons occupy narrow orbitals have properties that were harder to explain. In transition metals, electrons experience strong Coulomb repulsion due to their spatial confinement.The influence of an electron on the others is much pronounced here.

1.1 Strongly correlated systems

Strongly correlated materials form a class of heavy fermion compounds that include insulators and electronic materials. They exhibit unusual, yet technologically favourable electronic and magnetic properties such as metal-insulator transitions, half-metallicity and spin-charge separation. Many transition metal oxides having incompletely filled d- or f- electron shells with narrow energy bands fall under this category. They have electronic structures that are neither simply free-electron-like nor completely ionic, but a mixture of both. The interplay of d and f electrons' internal degrees of freedom-spin, charge, and orbital moment show eccentric ordering pheonomena at low temperatures.

This interplay makes strongly correlated electron systems extremely sensitive to small changes in external parameters, such as temperature, pressure, or doping. The dramatic effects can range from huge changes in the resistivity across the metal-insulator transition in Vanadium Oxide and considerable volume changes across phase transitions in actinides and lanthanides, to exceptionally high transition temperatures in superconductors with copper-oxygen planes. In materials called heavy fermion systems, mobile electrons at low temperature behave much massive than a free electron in a simple metal. Some of them display a very large thermo electric response while others are sensitive to magnetic fluctuations.

1.2 The theoretical challenge

Band theory, which captures the delocalized nature of electrons in metals, has been successful in explaining the behaviour of materials with weakly correlated electrons. Fermi liquid theory provides a conceptual picture of the spectrum of excitations in a solid which describes the transport of conduction electrons in momentum space. Failure of band theory was first noticed in insulators such as Nickel oxide and Manganese oxide, which have relatively low magnetic ordering temperatures, but large insulating gaps.

N. F. Mott illustrated that those insulators are better understood from a real-space picture of the solid as a collection of localized electrons bound to atoms with open shells.Adding and removing electron leaves it in an excited configuration. Since the spin and orbital angular momentum of the remaninig atoms scatter the excited configurations, these states propagate through a crystal incoherently and broaden to form bands, called the lower and upper Hubbard bands.

The richness of phenomena and extreme sensitivity to microscopic details make their experimental and analytical study difficult. Many models have been proposed to account for various phenomena that occur as a result of strong correlation. One of the simplest models of correlated electrons is the Hubbard model.

1.3 Operators

An operator maps vectors from one space **V** to vectors in another space **W**.

Mathematically, $\hat{\mathrm{A}} : \mathbf{V} \rightarrow \mathbf{W}$.

In linear algebra, the operator is a matrix operator. In Hilbert space, it can be a differential or integral operator. In Dirac's notation,

$$|g\rangle = A\,|f\rangle$$

1.3.1 Creation and Annihilation operators

In linear algebra, the creation and annihilation operators, collectively termed as ladder operators, increment or decrement a quantum number describing the state of a system. These operators are used in the formalism of quantum harmonic oscillator.

The quantum harmonic oscillator is characterised by the Hamiltonian

$$\hat{H} = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

We introduce the creation and annihilation operators

$$a = \frac{1}{\sqrt{2m\omega\hbar}}(m\omega x - ip) \tag{1.1}$$

$$a^\dagger = \frac{1}{\sqrt{2m\omega\hbar}}(m\omega x + ip) \tag{1.2}$$

The operators satisfy the commutation relation

$$\left[a, a^\dagger\right] = 1 \tag{1.3}$$

This gives rise to an infinite-dimensional Hilbert space spanningstates $|n\rangle$ for n=0,1,2,....∞

$$[a, H] = a$$

$$\left[a^\dagger, H\right] = -a^\dagger$$

The occupation number is defined as

$$n = a^\dagger a$$

We have the relations:

$$[n, a] = -a$$

$$\left[n, a^\dagger\right] = a^\dagger$$

1.3.2 Fermion Operators

Analogous to the harmonic oscillator operators, we have fermion creation and annihilation operators. In Hubbard model, there is a set of creation and destruction operators $c_{j\sigma}$ and $c\dagger_{j\sigma}$, where the index j refers to the spatial lattice site and σ refers to the electron spin (up or down).

First, we describe a vaccuum state, denoted by $|0\rangle$, which contains no fermions and is normalised.

$$\langle 0|0\rangle = 1$$

Then $c^{\dagger}_{j\sigma}$ is defined as to create a fermion of spin σ in the j-th single particel state by

$$c^{\dagger}_{j\sigma}|0...0\rangle = |0...010...0\rangle$$

where the numbers inside the ket indicate the occupation number of the various single particle states, and 1 is on the j^{th} site.
Pauli's exclusion principle prevents the occurence of two fermions in the same state, i.e.,

$$c^{\dagger}_{j\sigma}|0...010...0\rangle = 0$$

The anti commutation relation is given by

$$\{A, B\} = AB + BA \tag{1.4}$$

These operators obey the anti commutation relation

$$c_{j\sigma}c^{\dagger}_{l\sigma} = \delta_{j,l}\delta_{\sigma,\sigma'} \tag{1.5}$$

The following relations also hold true for the fermion ladder operators

$$\{\hat{a}, \hat{a}\} = \{\hat{a}^{\dagger}, \hat{a}^{\dagger}\} = 0 \tag{1.6}$$

i.e.,

$$\hat{a}\hat{a} = \hat{a}^{\dagger}\hat{a}^{\dagger} = 0 \tag{1.7}$$

1.4 The Many-Body Problem

The systems that we have in the physical world like electrons in a metal, molecules in a liquid,etc,.. comprise of many particles interacting with each other, giving rise to different phenomena. But, the particles in intricate motion are difficult to study. Hence, we make attempts to bring them to a single body problem just like two masses connected by a spring are treated as a single body by considering the centre of mass. But converting every physical problem to single body problem is practically impossible. So, we need to equip ourselves with techniques for handling many-body problems efficiently.

Many-body problem may be defined as the study of effects of interaction between bodies on the behaviour of a many-body system. It deals with general methods applicable to all many-body systems. To solve a many-body problem, we need to understand the problem first. The essential part is that there should be many bodies interacting with each other. A system of non-interacting bodies will not come under many body problem, rather it constitutes many one body problems. These interactions play a major role in determining the physical aspects of the system.

In many cases, many-body problems were solved by ignoring the interactions. Another approach used was the canonical transformation technique, which involves transforming the basic equations of the many-body system to a new set of coordinates in which the interaction term becomes small. These were to an extent, able to explain the behaviour of the system. But, the lack of a systematic method kept the many-body problem in its infant stage well up into 1950s. Emergence of quantum fiels theory opened new doors to deal with many-body problems.

1.5 Quasi Particles

Quantum field theory provided a new simple picture of matter in which systems of interacting real particles are described in terms of approximately non-interacting fictitious bodies called 'quasi particles' and 'collective excitations'. Consider two masses connected by a spring as shown in the figure.

Our system consists of two strongly coupled real bodies. The motion of each mass

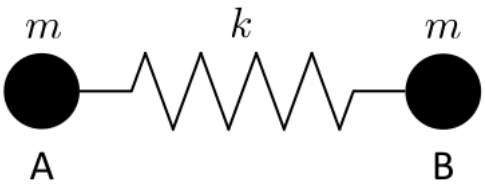


Figure 1.1: Two body system

is complicated due to interaction. We break up the complicated motion into two parts: motion of the centre of mass and motion about centre of mass. Centre of mass is an independetn body of mass 2m, and the latter part has a reduced mass of $\frac{m}{2}$. Thus, the system acts as if it was composed of two non-interacting fictitious bodies: centre of mass body and the reduced mass body.

The concept of quasi particle arises from the fact that when a real particle moves through the system, it pushes or pulls on its neighbours and thus becomes surrounded by a 'cloud' of agitated particles. The real particle plus its cloud forms the quasi particle. The presence of cloud makes the properties of quasi particle different from that of a real particle. The quasi particle has an effective mass and a lifetime. It is in an excited energy level of the many-body system, hence referred to as 'elementary excitation' of the system.

A model used to describe many-body effects in metalsis the electron gas model. It consists of a box containing a large number of electrons interacting with each other via Coulomb force. A uniform, fixed, positive charge background is present in the system to keep the whole system electrically neutral. In the ground state, electrons are spread out uniformly. Suppose we shoot a single, well-localised electron into the electron gas. The coulombic repulsions keep the other electron away from the new electron such that an empty space is created near the extra electron. The empty space has positive charge due to the background charge. Thus, the extra electron has lifted out electrons from the charge distribution in its vicinity, thereby creating holes.

This gives a picture of an extra electron surrounded by a cloud of constantly changing holes and lifted out electrons. This combination is termed as quasi electron. The positive hole cloud shields the negative charge of the electron. Hence, two quasi electrons whose hole cloud do not overlap, interact weakly due to shielding. This is why metals generally behave as independent of electron-electron interaction.

Quasi particles have an energy different from that of real particle due to the cloud surrounding it.

$$E = \frac{p^2}{2m*}$$

where m* is the effective mass.

$$E_{quasi} - E_{real} = E_{self} \tag{1.8}$$

where E_{self} is the self energy, which is the energy that a particle has due to the changes it caused by itself in its environment.

Chapter 2

Scope and Objectives

2.1 Scope

In most of the theories for physical system, we neglect the Coulombic repulsion between electrons, for convenience. But we see a deviation form the expected behaviour due to the presence of these interactions. Hence the models which take into account the repulsion between electrons is mandatory to have a full understandng of the behaviour of solids. Another criterion we follow in large is the nearest neighbour interactions. Although such models were successful, it is a vague approximation to make. So development of models which can explain interactions at a distance along with the coulombic repulsion can provide a major breakthrough in solid state physics.

It is believed that all information of an object that enters a black hole is lost forever. But, this is in contradiction with the basic law that information cannot be destroyed or created. This paradox could be solved by the holographic principle, which states that information in a higher dimension can be encoded as a hologram in lower dimension. A model for the same needed to be devised. But, in many theories that explain physical phenomena, we restrict the interactions to nearest neighbours. But it is not the case with the real systems. So, a model should be devised which can explain the system with long range interactions.

2.2 Objectives

- To get familiar with the fermionic creation and annihialtion operators
- To study the Hubbard model in detail
- To study in detail about the half filling of Hubbard model
- To understand the behaviour of Mott insulators
- To solve single and double site Hubbard model both analytically and present it graphically.
- To understand the treatment of many-body problems.
- To acquire an in-depth knowledge about Green's function and diagrammatic methods.
- To study about the principle of holography.
- To study about SYK model and its hamiltonian.
- To find the Green's function for both the models.

Chapter 3

Hubbard Model

3.1 The Model and its Hamiltonian

The Hubbard model, named after John Hubbard, is the simplest model of interacting particles in a lattice, with only two terms in the Hamiltonian: a kinetic term allowing for hopping of particles between sites of the lattice and a potential term consisting of an on-site interaction. The particles can either be fermions or bosons. It is the fermionic version that is the classic Hubbard model.

The standard Fermi-Hubbard model considers two species of fermionic particles which are confined in a lattice and move by hopping to the nearest neighbour sites. Electrons are allowed to move between localized states at lattice sites with spin $\sigma = \uparrow$ or $\downarrow$. Interaction takes place only among the particles at the same site. There exist various extended Hubbard models with longer range interactions, but we do not discuss them here. In optical lattices, the nearest neighbour hopping and on-site interaction are realized by having a sufficiently strong lattice potential and short range of the interparticle interaction.

The model was originally proposed in 1963 to describe electrons in solids and has been regarded as a model for high-temperature superconductivity. For electrons in solids,the Hubbard model can be considered as an improvement on the tight-binding model, which includes only the hopping term. For strong interactions, it gives qualitatively different behaviour from the tight-binding model and correctly predicts the existence of so-called Mott insulators, which are characterized by the strong repulsion between particles.

For the formulation of Hamiltonian, we must describe the motion and interactions of electrons in a solid. We begin with a lattice of atoms(sites) on which the electrons move and lattice vibration is neglected.

In a solid where electrons can move around, the electrons interact via a screened Coulomb interaction. The interaction term is zero if the site is empty or has only a single fermion, but has the value U if the site is doubly occupied. The interacton energy term in the Hamiltonian adds an energy U everytime it finds a doubly occupied site. In the simplest Hubbard model, the interactions between fermions in different sites are not taken into account. The kinetic energy describes the destruction of a fermion of spin σ on site l and its creation on site j (or vice-versa). Since wavefunctions die off exponentially, hopping is allowed only between the closest atoms in the lattice,which is symbolically represented as $\langle j, l \rangle$. Overlap of two wavefunctions on the pair of atoms determines the energy scale 't', which governs the hopping.

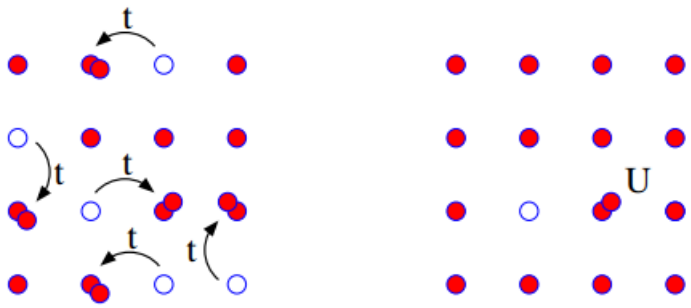


Figure 3.1: Representation of the terms in Hubbard Hamiltonian: the kinetic energy t and on-site interaction U

Thus, the Hubbard Hamiltonian is formalised as

$$\hat{H} = -t \sum \left(c^{\dagger}_{j\sigma} c_{l\sigma} + c^{\dagger}_{l\sigma} c_{j\sigma} \right) + U \sum n_{j\uparrow} n_{j\downarrow} - \mu \sum \left(n_{j\uparrow} + n_{j\downarrow} \right) \tag{3.1}$$

In the equation,the first term corresponds to chemical bonding and is known as 'hopping'. This term is a single particle interaction and hops an electron from one atom to a nearest neighbour with hopping matrix element t. The second term corresponds to the Coulomb repulsion between two electrons and is therefore a two-particle interaction.The third term corresponds to the chemical potential which tells how the energy

changes with the density of the system. At first look, the Hubbard model is all about the competition between chemical bonding and short range Coulombic repulsion. However, there is a third term in action: Pauli's exclusion principle. Electrons are fermions and so the many-particle wavefunction must be antisymmetric under interchange of any two electrons. The competititon between these three factors and the correlations between the electrons determine the behaviour of the model.

3.2 Single site Hubbard model

We study the simplest case of Hubbard model, that is, by considering just a single site. In other words, we set t=0 so that we have a collection of independent sites. Four possibilities are: site being empty $|0\rangle$, site having a single electron $|\uparrow\rangle$ or $|\downarrow\rangle$, and site being doubly occupied $|\uparrow\downarrow\rangle$,where $|0\rangle, |\uparrow\rangle, |\downarrow\rangle and |\uparrow\downarrow\rangle$ are eigenstates of Hamiltonian with eigenvalues 0, $-\mu, -\mu$ and U-2μ respectively.
The partition function is given by

$$Z = 1 + 2e^{\beta\mu} + e^{2\beta\mu - \beta U} \tag{3.2}$$

Total energy of the system can be written as

$$E = \frac{U e^{2\beta\mu - \beta U}}{1 + 2e^{\beta\mu} + e^{2\beta\mu - \beta U}} \tag{3.3}$$

Occupancy as a function of d-level at finite temperature is as follows:

$$\langle n \rangle = \frac{2(e^{\beta\mu} + e^{2\beta\mu - \beta\mu})}{1 + 2e^{\beta\mu} + e^{2\beta\mu - \beta U}} \tag{3.4}$$

3.3 Two site Hubbard model

We consider two sites each with 4 possibilities, giving rise to 16 combinations: both sites empty (1), both sites singly occupied (4), both doubly occupied (1), one is empty and the other is singly occupied (4), one is empty and the other is doubly occupied (2) and the case where one is singly and the other is doubly occupied(4). They result in the

energy eigenvalues respectively 0, -2μ, $2U-4\mu$, $-\mu$, $U-2\mu$ and $U-3\mu$. The partition function is given by

$$Z = \sum_{\alpha} \langle \alpha | e^{-\beta H} | \alpha \rangle \tag{3.5}$$

By substituting the eigenvalues and simplifying, we get the partition function as

$$Z = 1 + 4e^{\beta\mu}(1 + e^{\beta\mu}) + 2e^{2\beta\mu-\beta U}(1 + 2e^{\beta\mu}) + e^{4\beta\mu-2\beta U} \tag{3.6}$$

Occupation number can be found using the formula,

$$\langle n \rangle = Z^{-1} \sum_{n} \langle n | n e^{-\beta H} | n \rangle = Z^{-1} \sum_{n} n e^{-\beta H} \tag{3.7}$$

It can be calculated to be

$$\langle n \rangle = \frac{4\left[e^{\beta\mu}(1 + 2e^{\beta\mu}) + e^{2\beta\mu-\beta U}(1 + 3e^{\beta\mu}) + e^{4\beta\mu-2\beta U}\right]}{1 + 4e^{\beta\mu}(1 + e^{\beta\mu}) + 2e^{2\beta\mu-\beta U}(1 + 2e^{\beta\mu}) + e^{4\beta\mu-2\beta U}} \tag{3.8}$$

3.4 Mott Transition

A lattice can have a maximum of two electrons per site. So the case where the filling is one electron per site is termed as half filling. Studies of the Hubbard model is often studied at half filled case since it exhibits a lot of interesting phenomena. We modify the Hubbard Hamiltonian as

$$H = -t \sum_{\langle j,l \rangle \sigma} c^{\dagger}_{j\sigma} c_{l\sigma} + U \sum_{j} (n_{j\uparrow} - \frac{1}{2})(n_{j\downarrow} - \frac{1}{2}) - \mu \sum_{j} (n_{j\uparrow} + n_{j\downarrow}) \tag{3.9}$$

This new form differs from the original only by a trivial shift in chemical potential and an overall additive constant to the energy.

In simple words, the metal-insulator transition due to Coulomb interactions between electrons is called Mott transition. It provides an important foundation of how the electron-electron interactions lead to the insulating phase. Due to strong correlation, the original band would be split into two bands with energy gap of U, thus the system would behave as an insulator. To understand Mott transition which occurs in transition metals and layered organic compounds, it is necessary to know how electron change its state from being mobile to localized.

In normal materials, the electron-electron effects are negligible. In a Mott insulator, every atom has one orbital left at half-filling. In order to move an electron to another site, it would have to pair up two electrons, which is energetically unfavourable. In systems close to metal insulator transition, the physical properties change dramatically with the variation of parameters such as carrier concentration, temperature or external magnetic field. It is a dynamical transition and hence categorised as a first order phase transition.

In Mott insulators, coulomb interaction becomes comparable to Fermi energy, and the ground state of the system can undergo a change; the electrons become bound or localised. The material ceases to conduct. In a Mott insulator, the Mott gap is due to coulomb interactions, whereas, the bandgap in a band insulator is due to a chemical potential difference between sites or orbitals within a unit cell.

Mott transition directly addresses the competition between kinetic energy and correlation energy, that is, the wavelike and particlelike character of electrons in solid. The Coulomb interactions and the matrix elements that describe electron hopping from site to site are required to calculate the experimental phase diagram of materials near a Mott insulator.

3.5 Local moment

When the lattice has integer filling per unit cell, then the electrons can be mobile only if they have enough kinetic energy(t) to overcome the coulombic energy U. In the limit $t \ll U$, the electrons don't have kinetic energy, and a gap opens, leading to Mott insulating phase. This Mott gap is the energy an electron has to loose to overcome the coulomb repulsion and leave the lattice site. In simple words, it is the energy for an electron to hop to the nearest lattice site. This does not depend on the periodicity of the lattice.

In the ground state, each lattice site is singly occupied, and the electron occupying it behaves as aspin $\frac{1}{2}$ magnetic moment. These magnetic moments are localised to the lattice sites and gice rise to a fundamental quantity called "local moment". It is defined

as

$$\langle m^2 \rangle = \langle (n_\uparrow - n_\downarrow)^2 \rangle \tag{3.10}$$

Localised magnetic moments form the basic driving force of strong correlation.(**?**) Localised moments in metals have been accepted as an experimental fact without a rigorous study. Anderson tried to describe a quantum state of the metal in which such a moment exists. The atom is represented by an up- and a down-spin state available for electrons. If the atoms contain two or no elecetrons, then it has no magnetic moment. If it contains one electron, it does have a local moment. In the absence of hopping term, the eigenstates of the Hamiltonian are the localised orbitals on the individual site. When hopping is turned on, the wavefunction of the particel gets exponentially localised since its amplitude decays off exponentially.

Chapter 4

Green's Function

4.1 Definition and basic properties

The Green's function, named after the English mathematician and physicist George Green, is a powerful mathematical tool to solve linear differential equation with constraining boundary conditions. Green's function method enables the solution of a differential equation containing an inhomogeneous term, often called a source term, to be related to an integral operator containing the source.

As an elementary example, consider the Poisson's equaiton, which gives the potential ψ(r) generated by a charge distribution whose charge density is ρ(r).

$$-\nabla^2\psi(\mathbf{r}) = \frac{1}{\epsilon_0}\rho(\mathbf{r}') \tag{4.1}$$

The solution gives the Coulombic potential

$$\psi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{4.2}$$

The right hand side of the equation is an integral operator that converts ρ into ψ , and the kernel can be seen as the Green's function,

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi\epsilon}\frac{1}{|\mathbf{r}-\mathbf{r}'|} \tag{4.3}$$

i.e.,

$$\psi)(\mathbf{r}) = \int d^3r' G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}') \tag{4.4}$$

G(**r**,**r**') gives the contribution to ψ at the point **r** produced by a point source of unit magnitude at the point **r**'. In general, if D is a linear operator, f(x) is the desired solution and g(x) is the homogeneity source, then

$$Df(x) = g(x)$$

then the Green's function is defined as the solution of a differential equation with a delta homogeneity:

$$DG(x, x') = \delta(x - x') \tag{4.5}$$

and

$$f(x) = \int dx' G(x, x') g(x') \tag{4.6}$$

There are two types of Green's function: advanced and retarded Green's functions.If the final state of a system is given, we can find the inital state by retarded Green's function. We can calculate the response of a system after it is perturbed. On the other hand, if the initial state of the system is known, the final state can be predicted using advanced Green's function.

Properties

When D and its boundary conditions define the eigenvalue problem $D\psi = \lambda\psi$ with eigenfunctions $\phi(\mathbf{r})$ and corresponding eigenvalues λ_n, then

- G is symmetric.

$$G(r, r') = G(r', r) \tag{4.7}$$

- G has the eigenfunction expansion:

$$G(r, r') = \sum_n \frac{\phi_n^*(r')\phi_n(r)}{\lambda_n}$$

- G is continuous at r=r'.

$$G_{|r=r'+} = G_{|r=r'-}$$

- Derivative of G experiences a specific jump discontinuity at x=x'.

$$\frac{dG}{dr}_{|r=r'+} - \frac{dG}{dr}_{|r=r'-} = \frac{-1}{p(r')}$$

4.2 Propagators in Quantum Mechanics

In the classical scenario, the detailed description of a many-body system requires the position of each particle as a function of time $r_1(t), r_2(t)$,... In quantum case, it depends on the time dependent wave function of the system $\psi(r_1, r_2, ..., r_N, t)$. This is a difficult task to carry out. But it turns out that it is not necessary to know the detailed behaviour of each particle in order to find the important physical properties of the system, but rather just the average behaviour of one or two typical particles. The quantities that describe the average behaviour are called the propagators.

We can see that the Green's function takes a function at some space and evolve it to some other space. In fact, we say that Green's function is a propagator which propagates the particle from one point to another. Propagators play an important role in the treatment of many body problem. The reasons for their immense use is attributed as follows. First, they yield in a direct way the most important physical properties of the system. Second, they have a simple physical interpretation. Third, they can be calculated in a highly systematic and automatic way and appeals to one's physical intuition.

The idea is simple. We start with a system in its ground state say $|0>$. Then, we introduce (create) a particle in it which is used to probe the system. It makes interactions with the systems that might cause some excitations. Later, we remove (annihilate) the particle and check whether the system remains in ground state. Propagator is the probability amplitude that the system remains in ground state.

$$G(r_2, r_1) = \langle 0|(\text{particle annihilated})(\text{particle created})|0\rangle$$

There are one particle, two particle and zero particle propagators. One particle propagator can be defined as the probability amplitude that a particle will be observed at the point r_2 at time t_2 if it was intrtoduced into the system at a point r_1 in time t_1. Two particle propagator is the probability amplitude for observing one particle at (r_2, t_2) and another at (r_4, t_4) if they were put into the system at (r_1, t_1) and (r_3, t_3) respectively. Zero propagator or vacuum amplitude is the probability that no particle emerges from the system at t_2 when we introduce no particle at time t_2.

For instance, take the example of a metal that consists of a set of positively charged ions arranged so that they form a regular lattice or a lattice with some irregularities. An electron interacts with these ions by means of the Coulomb force. Here, the single particle propagator is the sum of the quantum mechanical probability amplitudes for all the possible ways the electron can propagate from point r_1 in the crystal to point r_2. It can propagate freely, or can intract with various ions. The total probability will be a sum of all these individual properties.

$$P(r_2, r_1) = P_o(r_2, r_1) + P_o(r_A, r_1)P(A)P_o(r_2, r_A) + P_o(r_B, r_1)P(B)P(r_2, r_B) + \mathrm{P}_o(r_A, r_1)P(A)P_o(r_A, r_A)P(A)P_o(r_2, r_A) + ... \tag{4.8}$$

where A,B, etc.. are the ions from which the electron interacts with, P_o denotes the free propagator and the term on left denotes the total propagator.

Now, assume that there is a particular affinity towards A and also the free propagators are all equal, i.e.,

$$P_o(r_2, r_1) = P_o(r_A, r_1) = P_o(r_2, r_B) = ... = c \tag{4.9}$$

Then,

$$P(r_2, r_1) = c + c^2P(A) + c^3P^2(A) + ..$$

$$P(r_2, r_1) = c[1 + cP(A) + c^2P^2(A) + ...]$$

$$P(r_2, r_1) = c[1 + cP(A) + (cP(A))^2 + (cP(A))^3 + ...]$$

This is an infinite geometric progression whose sum which can be summed as

$$P(r_2, r_1) = c\left[\frac{1}{1 - cP(A)}\right] \tag{4.10}$$

4.3 Electron's Green's function

The single electron Green's function is defined as the statistical expectation value of the product of fermion operators at different positions l and n and at different times t

and t'.

$$G(t,t') = -i\left\langle \theta(t-t')\left[C_l(t)C_n^\dagger(t')\right]\right\rangle \tag{4.11}$$

where $C_n^\dagger$ creates an electron at n^{th} site at time t' and C_l annihilates an electron on l_{th} site at time t; and, $\theta(t-t')$ is the Heaviside step function which takes the value 1 if t is greater than t', and 0 otherwise.

$$\left[C_l(t)C_n^\dagger(t')\right] = C_l(t)C_n^\dagger(t') - C_n^\dagger(t')C_l(t) \tag{4.12}$$

Then, (3.11) can be written as

$$G(t,t') = -i\theta(t-t')\left\langle\{C_l(t)C_n^\dagger(t')\}\right\rangle \tag{4.13}$$

where the operators obey anti commutation relation

$$\{C_l(t)C_n^\dagger(t')\} = C_l(t)C_n^\dagger(t') + C_n^\dagger(t')C_l(t) \tag{4.14}$$

4.4 Equation of Motion Technique

Equation of motion technique is used to find Green's function by determinng its time evolution. Let us start with Green's function.

$$G(t,t') = -i\theta(t-t')\left\langle\{C_l(t)C_n^\dagger(t')\}\right\rangle$$

$$\begin{aligned} i\delta_t G_{ij}(t,t') &= i(-i)\delta(t-t')\left\langle\{C_i(t)C_j^\dagger(t')\}\right\rangle - i\theta(t-t')\left\langle\{i\dot{C}_i(t)C_j^\dagger(t')\}\right\rangle \\ &= \delta(t-t')\delta_{ij} - i\theta(t-t')\left\langle\{[C_i,H](t),C_j^\dagger(t')\}\right\rangle \end{aligned} \tag{4.15}$$

This comes from the Heisenberg picture of operators.

$$i\frac{dC_i}{dt} = [C_i,H] + i\frac{\delta C_i}{dt} = [C_i,H] \tag{4.16}$$

Commutator in the LHS tells us that the dynamics of Green's function is fully determined by the Hamiltonian of the system. In this way, we will get a chain of Green's functions. However, for non-interacting systems, the commutator is a single fermion operator and ends at the first equation itself.

4.5 Recursive Green's function method

The recursive Green's functions use computationally efficient decimation techniques, simulating materials via effective layers. The idea of dividing the materials into layers and modelling it in a chain is the essence of recursive Green's function method. There are two kinds of decimation procedures used in these methods: surface technique, and an alternative version that stores information from the central sites.

Chapter 5

Diagrammatic Methods

5.1 Feynman Diagrams

There are two methods for calculating propagators. One is to solve the differential equations and the other is to expand the propagator in an infinite series and evaluate the series approximately. This can be done in a systematic, picturesque way with the help of Feynman diagrams.

The equation for propagator can be represented as a picture using Feynman diagrams.

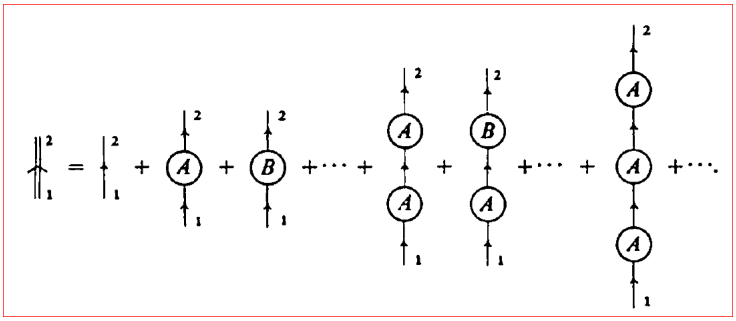
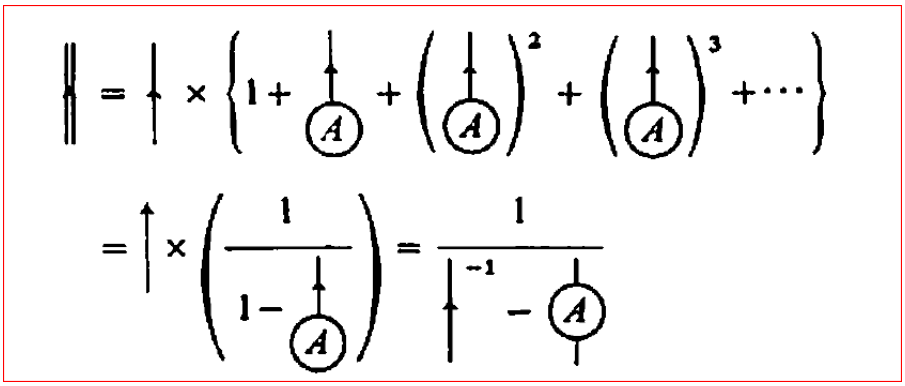


Figure 5.1: Pictorial representation of infinite series

Assuming that A dominates, the series can be approximated using partial summation.



5.2 Single particle propagator

Consider a system of many particles into which a particle is introduced at (r_1, t_1). The single particle propagator is the sum of the probability amplitudes for all the ways the particle can travel through the system from (r_1, t_1) to (r_2, t_2). It can have free propagation without interaction or it can emerge from the system after one or more interactions.

For first-order interactions, it follows the following steps:

- An extra particle enters the system at time t_1.
- It interacts with a particle at point r_2 and changes place with it.
- Extra particle leaves at time t_2

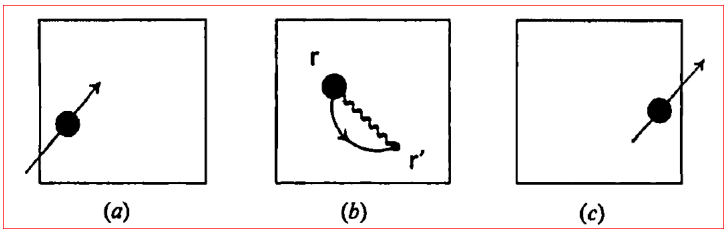


Figure 5.2: First order interaction

Second-order interaction follows the following steps:

- An extra particle enters the system at t_1.
- It interacts with a particle in the system at time t, lifting it out of its place, thus creating a hole in the system.

- The extra particle plus the 'hole' and the 'lifted-out particle' ('particle-hole pair') propagate through the system.
- The exrta particle interacts with the lifted-out particle at time t', knocking it back into the hole, thus destroying the particle-hole pair.
- The extra particle moves out of the system at time t_2.

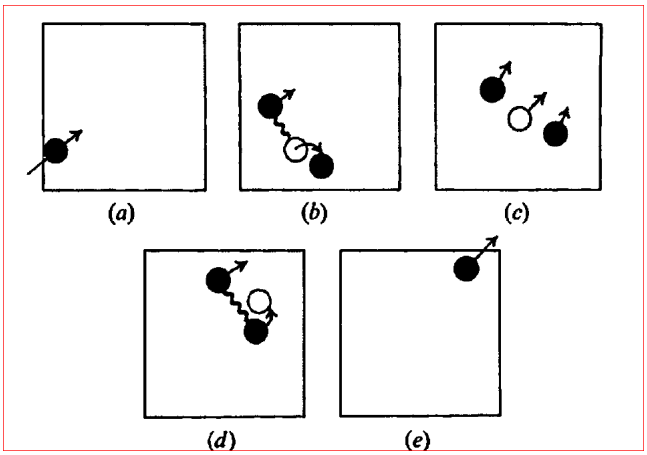


Figure 5.3: Second order interaction

To represent these diagrammatically, assume that the time increases in the upward direction.

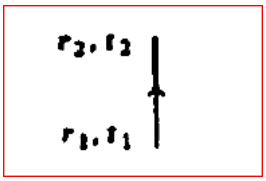


Figure 5.4: Propagator for particle moving freely from (r_1, t_1) to (r_2, t_2).

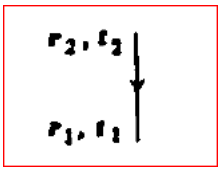


Figure 5.5: Propagator for hole moving freely from (r_1, t_1) to (r_2, t_2)

So, the total single particle propagator can be represented as the sum of all these Feynman diagrams.

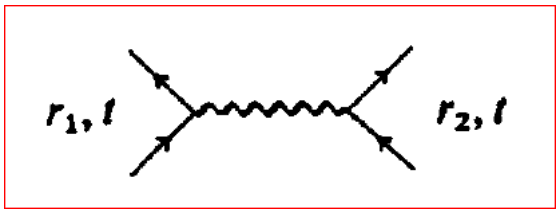


Figure 5.6: Propagator for a particle at r_1 interacting with a particle at r_2

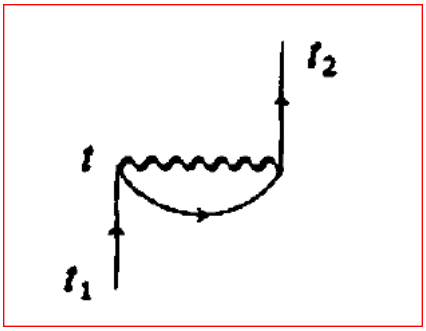


Figure 5.7: Open oyster diagram for first-order interaction

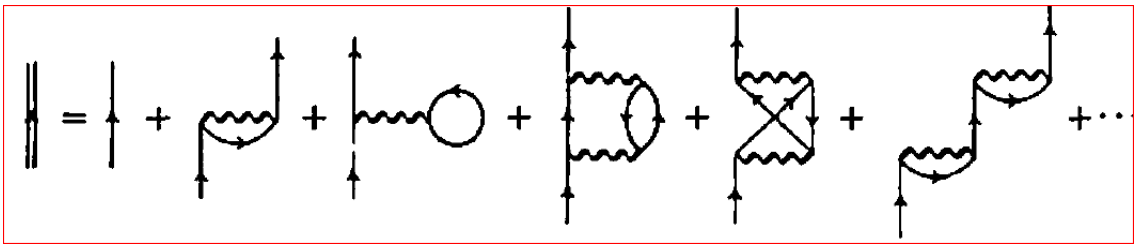


Figure 5.8: Feynman diagram for single particle propagator

5.3 Two particle propagator

Two particle propagator is the sum of probability apmlitudes for all the possible ways two particles can enter the system, interact with it and emerge at a later time.

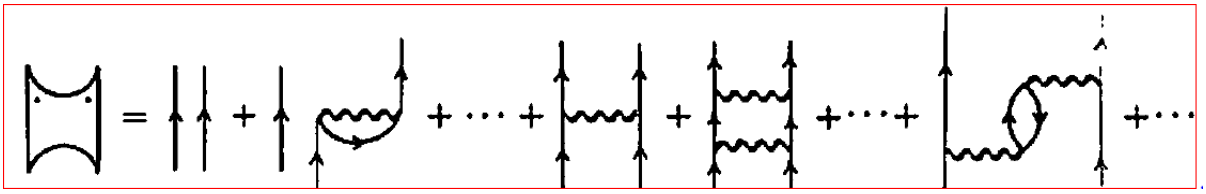


Figure 5.9: Feynman diagram for a two particle propagator

5.4 No particle propagator

This deals with the situation where a system has no extra particle in it. Interactions take place between particles inside the system. Finally, no particle emerges out of the system. It also goes by the term vacuum amplitude. It can be summarised in following steps:

- Vacuum.
- At t_1, interaction between two particles in the system cause them to be lifted out, forming holes.
- The two particle-hole pairs move freely through the system.
- Both pairs annihilate at t_2.
- Vacuum.

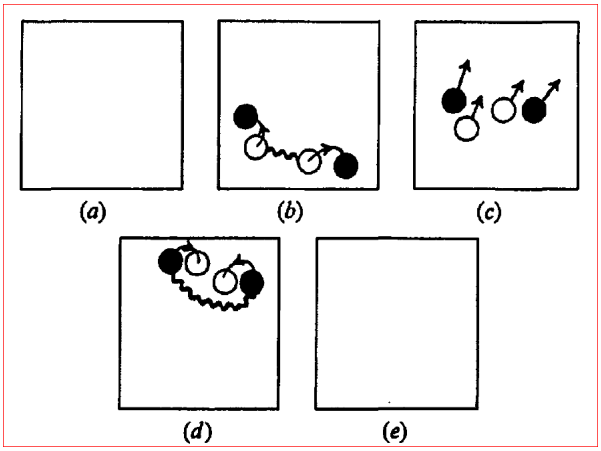


Figure 5.10: Representation of no particle propagator

5.5 Schwinger Dyson Equation

Schwinger Dyson Equations (SDEs), named after F. J. Dyson and J. S. Schwinger, are the equations of motion of Green's functions that describe the propagation and interactions of the fields of a theory. The full system of SDEs provide a complete description of the theory. The full system of SDEs provide a complete description of

the theory. The building blocks of SDEs are the Green's functions of the theory, which describe how a particle or field propagates and interacts. They are also called correlation functions or n-point functions.

$$G(\omega) = g(\omega) + g(\omega)\sum(\omega)G(\omega) \tag{5.1}$$

$$G(i\omega)^{-1} = -i\omega - \sum(i\omega) \tag{5.2}$$

where $g(\omega)$ is the undressed Green's function corresponding to free propagation; and $G(\omega)$ is the dressed Green's function corresponds to interactions. $\sum(\omega)$ denotes the self energy which accounts for the effects of self-consistent interactions. For interacting problems, the determination of a consistent self-energy is a challenging problem. Schwinger Dyson equations form an infinite series of couple integral equations. In order to solve them, system must be truncated using some constraints. These equations can be diagrammatically represented as follows:

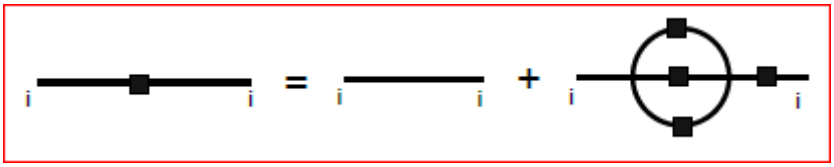


Figure 5.11: Schwinger Dyson Equation: Diagrammatic representation

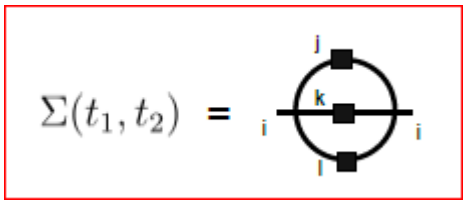


Figure 5.12: Self energy

Chapter 6

A Brief Introduction to Holography and Sachdev Ye Kitaev (SYK) Model

6.1 Black hole and Information Paradox

Black hole is a region of space-time from which gravity prevents anything, including light from escaping. The theory of general relativity predicts that a suffciently compact mass will deform space-time to form a black hole. It absorbs all the light that falls into it and reflects nothing, just like a perfect black body. Around a black hole, there is a mathematically defined surface called an *event horizon* that marks the point of no return.

In 1970s, Stephen Hawkings stated that when an object crosses the event horizon of a black hole, it is continuously dragged to the centre of the blackhole where it gets crushed. Every object contains a specific arrangement of matter, and hence some kind of information. When it is crushed, the information is destroyed.

If a black hole can evaporate, a portion of the information it contains is lost forever. The information contained in thermal radiation emitted by a black hole is degraded; it does not recapitulate information about matter previously swallowed by the black hole. The irretrievable loss of information conflicts with one of the basic postulates of quantum mechanics. According to Schrodinger equation, physical systems that change over

time cannot create or destroy information, a property known as '*unitarity*'. This apparent contradiction between general relativity and quantum mechanics is the information paradox.

A black hole in two dimensional space-time is given by 1 time and 1 space dimensions. It is a strongly chaotic system and should be described by a Hamiltonian. Its time evolution is given by a matrix. To resolve the information paradox, holographic principle has been introduced.

6.2 Holography

Studies showed that informartion paradox can be explained by holographic principle or holography. It states all the things falling inside a black hole were somehow captured in the horizon itself. So, the information was not lost, it is encoded on the surface of the black hole like a hologram. The information can be later retrieved in a chaotic form through the radiation released during quantum evaporation of the black hole.

The basic idea is that information carried by all the 3D objects in the world would be carried by some 2D surface that surround us; and we are just the holographic projection of that. The information is stored on some kind of holographic film on the edge of the universe. Indeed, it is now believed that all of the information contained within our Universe is nothing more than a 3D projection of the information stored on a 2D membrane at the edge of the cosmos.

The information contained in a black hole is not proportional to its volume, but its surface area. If we add one bit of information, the black hole grows by one square of planck unit. Holography is a convenient mathematical tool for us to solve higher dimensional problems by reducing it to lower dimensions. This principle hepls in explaining quantum gravity. If you make mapping from the three-dimensional Universe to two-dimensional surface, it was found that the gravity disappears. So, our 3D Universe with gravity might be equal to a 2D Universe without gravity. So, if we could do calculations on the 2D structure which are normalisable and do not lead to infinity, then we can map it back to the 3D structure to explain about our Universe.

6.3 Motivation: SY model

The model for holography is greatly inspired by Sachdev Ye (SY) model, a random quantum spin system originally introduced to describe a quantum Heisenberg magnet with random infinite-range interactions. The original SY model is given by the Hamiltonian

$$H = \frac{1}{\sqrt{M}} \sum_{j,k=1}^{N} J_{j,k} \mathbf{S}_j . \mathbf{S}_k \tag{6.1}$$

where the couplings $J_{j,k}$ are independent random variables drawn from a Gaussian distribution, each with the same variance and a mean zero. $\mathbf{S}_j$s are spin operators at lattice sites which may take the value $\frac{1}{2}, 1, \frac{3}{2}, 2, ...$ The coupling constant J decreases as the distance between spins increases, but is a constant for nearest neighbours.

Quantum magnets are spin systems in which the spins interact via quantum mechanical exchange interactions. There are two possible ground states of a random quantum magnet. One is called spin-glass, which is a state with magnetic long range order with $\langle S_i \rangle \neq 0$. Another one is called spin fluid which is a quantum disordered state with $\langle S_i \rangle = 0$. At zero temperature, there is a transition from spin fluid to the magnetically ordered spin glass phase.

6.4 SYK model

6.4.1 Hamiltonian

Sachdev Ye Kitaev (SYK) model is a model for holography which relates quantum mechanical systems to quantum gravity systems. So, what we consider is a quantum mechanical system living in 0 space and 1 time and is related to quantum gravity problem in 1 space and 1 time. This model is for the systems that share some properties with black holes. SYK model deals with the quantum mechanics of N fermions, which satisfy commutation relation.

$$\{\chi_i, \chi_j\} = \delta_{ij} \tag{6.2}$$

$\chi_i s$ are L x L matrices and are called Majorana fermions. To formulate the Hamiltonian for the model, we build a matrix as a sum over all possible products of four of

these fermion operators.

$$H = \frac{1}{4!} \sum_{i<j<k<l} J_{ijkl} \chi_i \chi_j \chi_k \chi_l \tag{6.3}$$

where J_{ijkl} are couplings drawn from a random Gaussian distribution.

$$P(J_{ijkl}) = \exp\left(\frac{-N^3 J_{ijkl}^2}{12 J^2}\right) \tag{6.4}$$

J defines the ensemble with mean

$$\langle J \rangle = 0$$

and the variance

$$\langle J^2 \rangle = \frac{3! J^2}{N^3} \tag{6.5}$$

Kitaev argued that it behaved in a way that was characteristic of systems which are holographic to quantum gravity.

Based on the similarities between the two and four point functions of 1 + 1 dimensional Schwarzchild black hole and those of the SYK model, this model has been proposed as a holographic dual to Schwarzchild black hole.

6.4.2 Majorana Fermions

All the matter in the Universe are made up of particles called fermions. Physicists know three kinds of fermions: Dirac, Weyl and Majorana fermions. Dirac fermions have mass, and is the class to which most fermions that make up matter belong. Weyl fermions are fermions that have no mass, and have only been found in lurking in solids with strange properties. A Majorana fermion is its own anti particle, i.e., when one Majorana fermion bumps into another Majorana fermion, they will both vanish.

Majorana fermions are electrically neutral elementary particles of matter first theorized by Ettore Majorana in 1937. Since they are their own anti particles, researchers believe that Majorana fermions may be useful in creating quantum computers because when moved, they are believed to have a property which allows them to remember their former position.

The difference between Dirac fermion and Majorana fermion can be mathematically expressed using creation and annihilation operators of second quantisation. The creation

operator creates a fermion in the site, whereas, an annihilation operator annihilates it. For Dirac fermion, the fermion operators are different, while, its same for Majorana fermions. The ordinary fermionic operators annihilation and creation operators a and $a^\dagger$ can be written in terms of two Majorana operators γ_1 and γ_2 as

$$a = \frac{\gamma_1 + i\gamma_2}{\sqrt{2}}$$

$$a^\dagger = \frac{\gamma_1 - i\gamma_2}{\sqrt{2}}$$

These operators satisfy the anti-commutation relation

$$\{\gamma_i, \gamma_j\} = \delta_{ij}$$

Since particles and anti particles have opposite conserved charges, Majorana fermions have zero charge. They cannot possess intrinsic electric or magnetic moments.

Chapter 7

Results and Discussion

7.1 Hubbard Model

7.1.1 Analytical solution of Green's function

The single particle Green's function is defined as the statistical expectation value of the product of fermion operators at different positions 'l' and 'n' and different times t and t'.

$$G_{ln}(t,t') = \left\langle c_l(t) c_n^\dagger(t') \right\rangle \tag{7.1}$$

where $c_n\dagger$ creates an electron on n-th site at time t' and c_l annihilates an electron on l-th site at time t.

$$\left[c_l(t) c_n^\dagger(t')\right] = \theta(t-t') c_l(t) c_n^\dagger(t') - \theta(t'-t) c_n^\dagger(t') c_l(t) \tag{7.2}$$

where θ denotes the Heavyside step function which takes the value 0 if the argument is negative, and the value 1 if otherwise.
According to Heisenberg's representation, the operator evolves from t' to t as

$$\hat{c_l}(t) = e^{i\hat{H}(t-t')} \hat{c_l}(t') e^{-i\hat{H}(t-t')} \tag{7.3}$$

Then, the Green's function becomes

$$G_{ln}(t,t') = -i\theta(t-t') \left\langle \{\hat{c_l}(t), \hat{c_n}^\dagger(t')\} \right\rangle \tag{7.4}$$

$$G_{nn}(t,t') = \frac{-i}{Z} \theta(t-t') \sum_{nm} |\left\langle n|c_n(t')|m\right\rangle|^2 e^{-i(\epsilon_m - \epsilon_n)(t-t')} (e^{-\beta\epsilon_n} + e^{-\beta\epsilon_m}) \tag{7.5}$$

When the system is at equilibrium or when the Hamiltonian is independent of time, it is convenient to represent Green's function in energy domain than in time domain. In this case,

$$G(\omega) = \frac{1}{Z}\sum_{mn}\frac{|\langle n|c_n(t')|m\rangle|^2}{\omega - (\epsilon_m - \epsilon_n) + i\eta}(e^{-\beta\epsilon_n} + e^{-\beta\epsilon_m}) \tag{7.6}$$

For non-interacting Hamiltonian, the poles of the Green's function corresponds exactly to the eigen energies.

7.1.2 Single site

The following figures show occupation number as a function of chemical potential. At half filling, $\rho = 1$. We do not fill the site with a second particel $\rho = 2$ until μ jumps by U. Presence of one fermion on a site blocks the addition of a second. There is a flat region form $\mu = -U/2$ or $U/2$. If we reach a region where no more energy levels exist, μ has to take a jump to the bottom of the next band. The jump in μ reflects the existence of a gap.

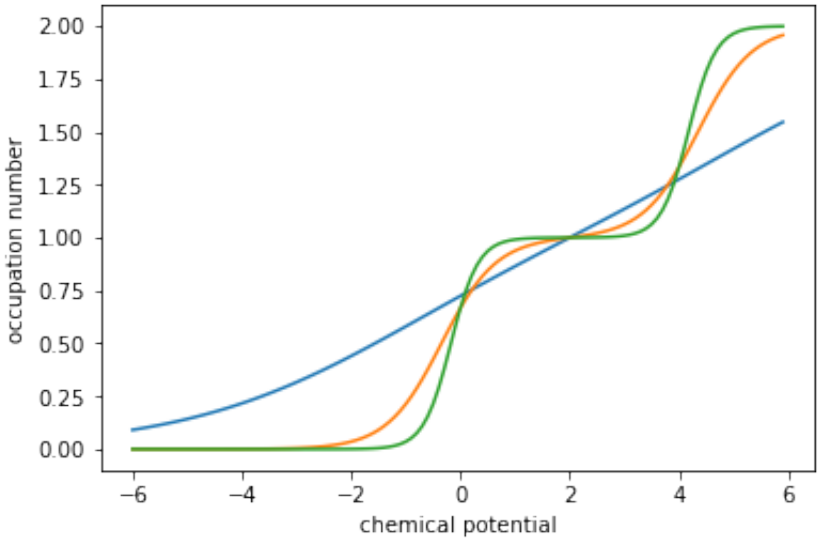


Figure 7.1: Plot of $\langle n\rangle$ v/s μ at U=4 for different temperatures

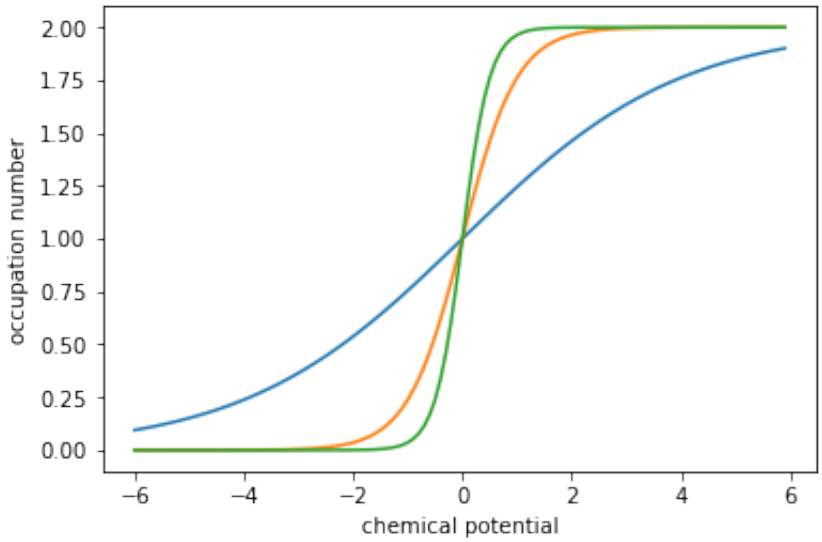


Figure 7.2: Plot of $\langle n \rangle$ v/s μ at U=0 for different temperatures.

7.1.3 Two site

The following figures show the occupation number of a two site Hubbard model at different temperatures. It can be observed that at lower temperatures, it shows insulating behaviour. All graphs are plotted at U = 4.

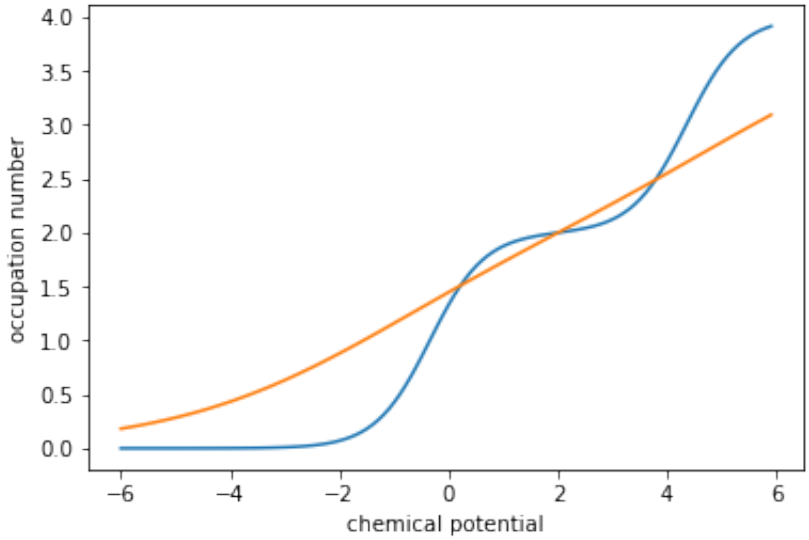


Figure 7.3: Plot of $\langle n \rangle$ v/s μ for $\beta = 2$ and $\beta = 0.5$.

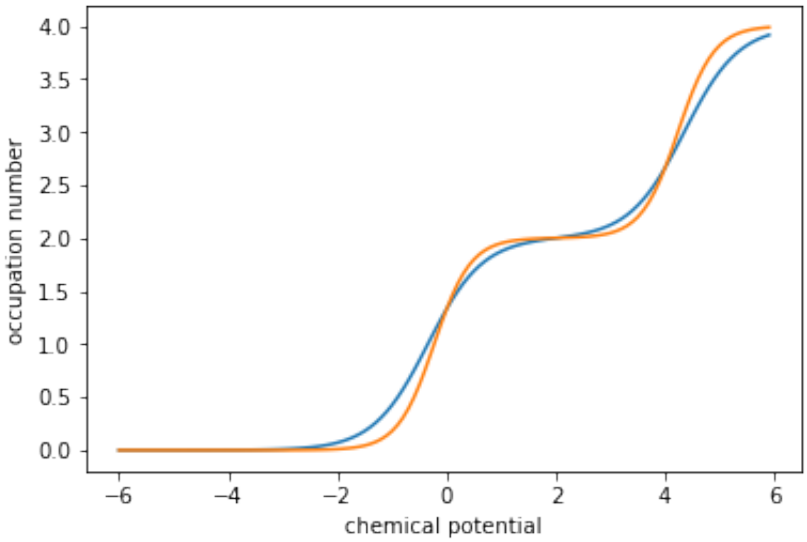


Figure 7.4: Plot of $\langle n \rangle$ v/s μ for $\beta = 0.5$ and 3.

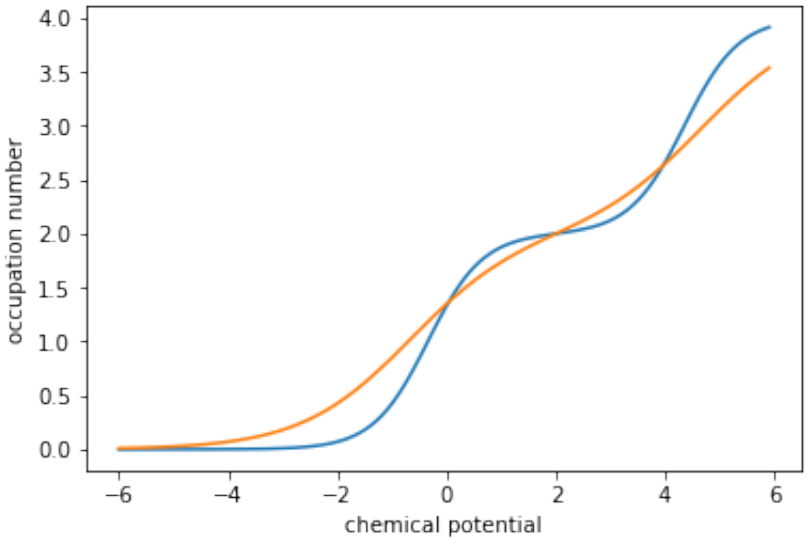


Figure 7.5: Plot of $\langle n \rangle$ v/s μ for $\beta = 0.5$ and 1.

7.2 SYK Model

7.2.1 Density of states of infinite chain

We define a linearly spaced vector of energies and evaluate the undressed Green's function. Then, we find the Green's function for the semi-infinite chain using the analytic expression and determine the infinite chain Green's function by joining two semi-infinite chains. From, the imaginary part of the Green's function, density of states can be calculated using the formula

$$\rho_1(\omega) = \frac{-1}{\pi} ImG_{11}(\omega) \tag{7.7}$$

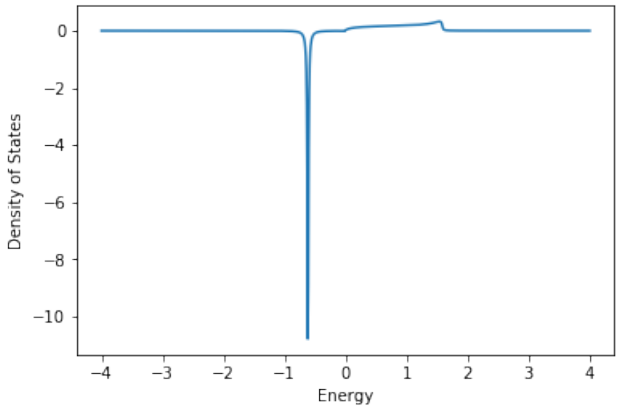


Figure 7.6: Density of states of an infinite chain

7.2.2 Analytical solution of Green's function

Green's functions are also called n-point functions. Wick's theorem states that for non-interacting particles, higher order Green's function involving more than one particle can be factorized into products of single-particle Green's functions. The free two-point function for the SYK model is given by

$$G_o(t_1, t_2) = -\langle T\chi_i(t_1)\chi_j(t_2)\rangle$$

where T denotes the time ordering of operators, which can be expanded using Heaviside step function as

$$T\chi_i(t_1)\chi_j(t_2) = \theta(t_1 - t_2)\chi_i(t_1)\chi_j(t_2) - \theta(t_2 - t_1)\chi_j(t_2)\chi_i(t_1)$$

We use the relation between Heaviside step function and signum function

$$sgn(t_1 - t_2) = 2\theta(t_1 - t_2) - 1$$

$$G_o(t_1 - t_2) = \frac{-1}{2} sgn(t_1 - t_2)\delta_{ij} \tag{7.8}$$

$$G(t_1, t_2) = G_o(t_1, t_2) + J^2 \int dt_a dt_b G_o(t_1, t_a) G(t_a, t_b)^3 G(t_b, t_2)$$

$$= \mathrm{sgn}(t_1 - t_2)\tfrac{1}{2} - \tfrac{J^2}{2} \int dt_a dt_b sgn(t_1 - t_2) G(t_a, t_b)^3 G(t_b, t_2) \tag{7.9}$$

Self energy is given by

$$\sum(t_1, t_2) = J^2 G(t_1, t_2)^3 \tag{7.10}$$

Schwinger Dyson equation gives

$$G(i\omega)^{-1} = -i\omega - \sum(i\omega)$$

In the limit where ω tends to zero,

$$-1 = G(i\omega) \sum(i\omega) \tag{7.11}$$

Let us take a function of the form,

$$G(t_1, t_2) = c|t_1 - t_2|^x sgn(t_1 - t_2) \tag{7.12}$$

$t_1 = \mathrm{t}$ and $t_2 = 0$ gives

$$G(t, 0) = c|t|^x sgn(t)$$

Fourier transform of G(t,0),

$$G(i\omega) = \int dt e^{i\omega t} c|t|^x sgn(t) \tag{7.13}$$

We know that the Gamma function is defined as

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx \tag{7.14}$$

$$G(i\omega) = 2iC sgn(\omega) cos\left(\frac{\pi x}{2}\right) \Gamma(1 - x)|\omega|^{x-1} \tag{7.15}$$

$$\sum(i\omega) = J^2 \int dt e^{i\omega t} C^3 |t|^{3x} sgn(t) \tag{7.16}$$

$$\sum(i\omega) = 2iJ^2 C^3 sgn(\omega) cos\left(\frac{3\pi x}{2}\right) \Gamma(1 - 3x)|\omega|^{3x-1} \tag{7.17}$$

Using this, we can write

$$-1 = -4J^2C^4 cos\left(\frac{\pi x}{2}\right) cos\left(\frac{3\pi x}{2}\right) \Gamma(1-x)\Gamma(1-3x)|\omega|^{4x-2}$$

Put x= $\frac{1}{2}$ and simplify to obtain

$$C = -\left(\frac{1}{4\pi J^2}\right)^{\frac{1}{4}}$$

From here, Green's function can be obtained as

$$G(t_1, t_2) = -\left(\frac{1}{4\pi J^2}\right)^{\frac{1}{4}} \frac{1}{\sqrt{t_1 - t_2}} sgn(t_1 - t_2) \tag{7.18}$$

7.2.3 Density of states of semi-infinite chain

We use the recursive method to evaluate the surface density of states of a semi-infinite linear chain. Effective Green's function of site 1,2 and 3 can be calculated from

$$\tilde{g_1} = \frac{g_1}{1 - g_1 t g_a t^*}$$

$$\tilde{g_2} = \frac{g_2}{1 - g_2 t^* g_a t - g_2 t g_b t^*}$$

$$\tilde{g_3} = \frac{g_3}{1 - g_3 t^* g_b t}$$

Recursive decimation procedure is run for 16 steps. We renormalize the hoppings and undressed Green's functions, carrying the decimation.

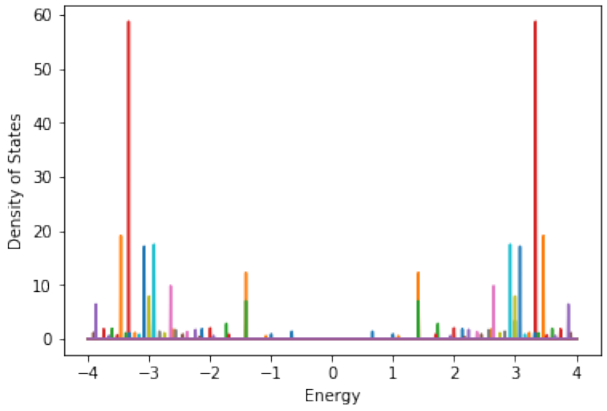


Figure 7.7: Density of states of the surface site

7.2.4 Density of states of three-site chain

We use the equation of motion technique to find the density of states of a three-site linear chain with nearest neighbour hoppings. Green's function of sites can be found using the relations

$$G_{11} = \frac{g_1}{1 - (\frac{g_1 t g_2 t^*}{1 - g_2 t g_3 t^*})}$$

$$G_{22} = g_2 + g_2 t^* G_{12} + g_3 t G_{32}$$

$$G_{33} = \frac{g_3}{1 - (\frac{g_3 t^* g_2 t}{1 - g_2 t^* g_1 t})}$$

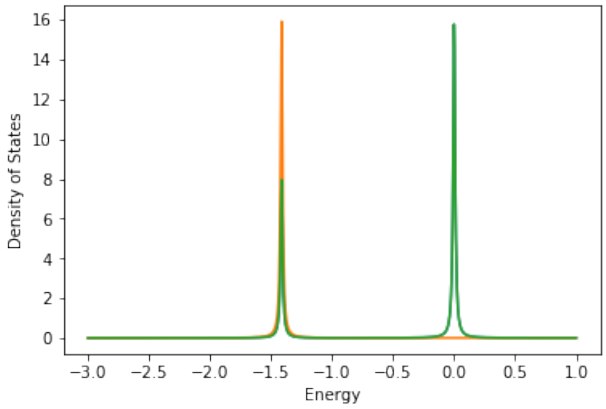


Figure 7.8: Density of states of a three site linear chain

The orange line shows the density of states of site 2 (central site), having only one peak. It is different from that of the external sites 1 and 3, which are shown by the green line, which has 2 peaks.

Chapter 8

Summary and Conclusion

In third semester, the project strated with a detailed understanding of quantum harmonic oscillator, ladder operators and fermion operators. Partition function, occupation number and Hubbard Hamiltonian were studied in detail. The model was solved for single site and two site. Green's function for the model was found analytically. Through this study, we could show that certain materials which are conventionally predicted as metals can show insulating behavoiur, a property known as Mott transition.

In this semester, we started off with addressing many body problems and finding a way to handle them easily via Feynman diagrams. Green's function is discussed in detail and the idea of propagators is introduced. Infromation paradox is reviewed and holographic principle is put forward as a solution. Sachdev Ye Kitaev model, which is proposed for holography is studied in detail. An in-depth study about the Hamiltonian and the operators included were also carried out. Green's function for the model is determined analytically. Through this work, it can be concluded that SYK serves as a model for holography and can be applied to system with infinite range interactions.

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