QUANTUM ERROR CORRECTING CODES

Project Report II

submitted in partial fulfillment for the degree of

MASTER OF SCIENCE
IN
PHYSICS

by

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Under the Guidance of

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DECLARATION

I hereby declare that the report of the P.G. project work entitled "QUANTUM ERROR CORRECTING CODES" which is submitted to National Institute of Technology Karnataka, Surathkal, in partial fulfilment of the requiremet for the award of the Degree of Master of Science in the Department of Physics, is a bonafide report of the work carried out by me. The material contained in this report has not been submitted to any University or Institution for the awards of any degree. In keeping with the general practice in reporting scientific observations, due acknowledgement has been made whenever the work described is based on the findings of other investigators.

Place: NITK Surathkal
2020

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CERTIFICATE

This is to certify that the project entitled ”QUANTUM ERROR CORRECTING CODES” is an authenticated record of work carried out by Aswanth K, Reg. No 186026 in partial fulfilment of the requirement for the award of the Degree of Master of Science in Physics which is submitted to the Department of Physics, National Institute of Technology, Karnataka during the period 2018-2020.

Dr Deepak Vaid
Project Advisor

Chairman- DPGC

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Place: NITK Surathkal
Date: 2020

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Abstract

One of the main roadblocks in the development of Quantum Computers is its vulnerability for accumulating errors through interaction with the environment. This is why we need to isolate and cool them to cryogenic temperatures. Even then these interactions cannot be completely nullified and the qubits will eventually decohere. Quantum Error Correction (QEC) can help us detect and correct such errors and make our Quantum Computation fault-tolerant under certain circumstances. QEC's are loosely based on the Classical Error-Correcting (CEC) codes. CEC's use redundant information to detect and correct errors. But Quantum Information cannot be copied due to the No- Cloning theorem. So QEC's encodes the information onto a highly entangled state and detects errors by making suitable measurements on this entangled system without disturbing the superposition of the state. A general formalism called the Stabiliser formalism has been developed to study different types of QEC's. In this project, the formalism of Quantum Error Correction is studied.The graphical language of tensor network in the context of Open Quantum Systems is also explored.Finally code for simulating arbitrary surface code error correction circuits was realised in Qiskit.

Contents

1 Introduction

Quantum computation is a distinctively new way of harnessing nature. It will be the first technology that allows useful tasks to be performed in collaboration between parallel universes.

– David Deutsch

The field of Quantum Computing has made enormous progress in the last 50 years. The theory of computation has made enormous leaps from an abacus of the Babylonians to Supercomputers capable of simulating black hole collisions. The present digital computers which are based on classical logic, even though extremely powerful, turn out to be inefficient in simulating real quantum systems, as observed by Richard Feynman [13]. In the paper, he concludes that to efficiently simulate a quantum system the computer should itself be a quantum system. This is because the states of quantum systems (e.g, a quantum magnet with N spins) are inherently probabilistic and requires exponentially(2^N) many 'amplitudes' to be specified for a given size. A quantum computer is such a system that leverages quantum phenomenon like entanglement and superposition to provide exponential speedups to problems.
The main difference between a classical and quantum computer is their state space. A classical computer stores information as strings of binary variables $\in \{0, 1\}$ called bits and perform computations using logical operators e.g AND, OR, NAND. A quantum computer on the other hand stores information as a vector living in a Hilbert space and performs computations via Unitary Transformations on that Hilbert space. The analogue of bits in quantum computing is a *qubit* which is a normalized vector in $\mathbb{C}^2$. Unitary operators act linearly on this vector and this can be thought of as a computation happening parallelly on each basis state of the qubit. This is where QC derives its exponential speedups for various kinds of problems like,

- Simulation of Chemistry
- Machine Learning algorithms with large feature vectors and training sets
- Finding Prime Factors of Large Numbers
- Searching for elements in a list

Presently Quantum Computing is in its infancy. We are in the Noisy Intermediate Scale Quantum Computing(NISQ) Era. The number of Qubits in present Quantum Computers ranges from 10 - 50 Qubits. Many of the interesting applications of QC requires thousands of qubits. One of the main roadblocks in achieving this is noises, accumulated due to interaction with the environment. This is why we need to isolate and cool them to cryogenic temperatures. Even then these interactions cannot be completely nullified. Quantum Error Correction (QEC) can help us detect and correct such noises and make our quantum computation fault-tolerant under certain circumstances. Therefore, only with the development of scale-able QEC's can QC realize its full potential. QEC's are loosely based on the Classical Error-Correcting (CEC) codes. CEC's use redundant information to detect and correct errors. But, Quantum Information cannot be copied due to the No-Cloning theorem. So, QEC encodes the information onto a highly entangled state and detects errors by making suitable measurements on this entangled system, without disturbing the superposition of the state. A general formalism called the Stabiliser formalism has been developed to study different types of QEC's. In this project, the formalism of Quantum Error Correction, the graphical language of tensor networks, and its applications are studied.

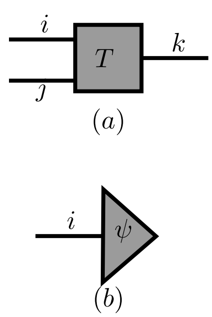
1.1 Tensor Networks

Tensors are a fundamental mathematical object used throughout physics, maths and computer science. It is an n-dimensional array of real or complex numbers. But what makes tensors useful is that it represents multi-linear mappings between vector spaces. It is a function of multiple parameters such that it is linear in each parameter. Interpreting tensors as mappings can introduce the notion of coordinate Independence which Einstein used to formulate his General Theory of Relativity in a beautiful geometric way. Tensor Networks are a collection of tensors joined by contractions. These networks can be studied and visualised using an elegant graphical language developed by Roger Penrose, which are discussed below. It was later used by David Deutsch to develop the quantum circuit model of quantum computation.

1.1.1 Penrose Graphical Notation

This notation, developed by Penrose, has the advantage over algebraic representations because it can provide a rich visual intuition for otherwise hard concepts. Different shapes can denote different types of tensors. Open legs pointing to the left denotes upper indices (kets) and open wires pointing to right denotes lower indices (bras). A few examples are shown below:

[1]



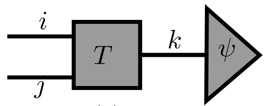
$$= \sum_{ijk} T_k^{ij} | \, i \, \rangle | \, j \, \rangle \langle \, k \, |$$

$$= \sum_i \psi^i | \, i \, \rangle$$

Figure 1: (a):(2,1)-Tensor. (b):vector

[1] All images by the author unless credited

Figure (a) represents a Rank-3 Tensor in the space $\mathscr{H}_i \otimes \mathscr{H}_j \otimes \mathscr{H}^k$. Figure (b) is a Rank-1 Tensor living in $\mathscr{H}_i$



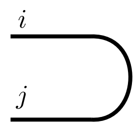
(c)

$$= \sum_{ij} \left(\sum_k T^{ij}_k \psi^k \right) |\, j\,\rangle|\, i\,\rangle$$

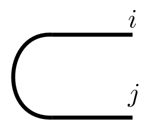
Figure 2: Contraction between a (2,1) Tensor and a Vector

When wires are joined together they denote contraction between indices of a tensor. For example in fig 2.(c) index-k is contracted between T^{ij}_k and ψ^k.

$(d)\ \delta^i_j$



$(e)\ \delta^{ij}$



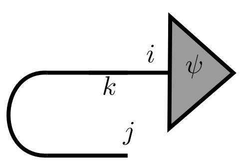
$(f)\ \delta_{ij}$

$$(d)\quad \delta^i_j = \sum_{ij} \delta^i_j |i\rangle\langle j| = \sum_i |i\rangle\langle i| = |0\rangle\langle 0| + |1\rangle\langle 1| + ... + |d\rangle\langle d|$$

$$(e)\ \delta^{ij} = \sum_{ij} \delta^{ij} |i\rangle|j\rangle = \sum_i |i\rangle|i\rangle = |0\rangle|0\rangle + |1\rangle|1\rangle + ... + |d\rangle|d\rangle$$

$$(f)\ \delta_{ij} = \sum_{ij} \delta_{ij} \langle i|\langle j| = \sum_i \langle i|\langle i| = \langle 0|\langle 0| + \langle 1|\langle 1| + ... + \langle d|\langle d|$$

Figure 3: (d) is the *identity wire* (e) is called the *cup* and (f) is called *cap*. cup and caps are used to raise and lower indices. They play the role of metric tensor in relativity.

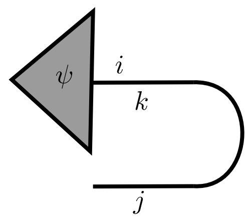


(g)

$$= \sum_{jk} \delta_{kj} \langle j|\langle k| \sum_i \psi^i |i\rangle = \sum_j \langle j| \left(\sum_{ik} \delta_{kj} \psi^i \langle k|i\rangle \right)$$

$$= \sum_j \langle j| \left(\sum_{ik} \delta_{kj} \psi^i \delta^i_k \right) = \sum_i \psi^i \langle i|$$

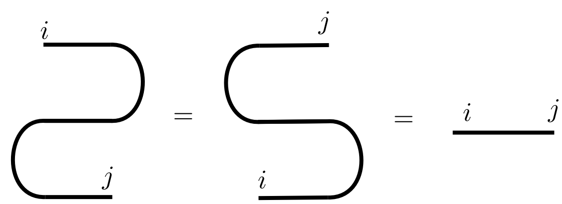
Figure 4: (g) shows a contraction between a ket and a cap which converts it to a bra.



$$= \sum_i \psi^i |i\rangle$$

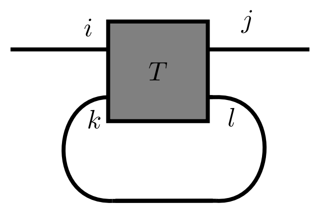
(h)

Figure 5: (h) shows a contraction of a bra with a cup which converts it to a ket



(i)

Figure 6: (i) is the Snake Equation



$$= trace_{k,l}\left(T^{ik}_{jl}\right) \;=\; \sum_k T^{ik}_{jk}$$

(k)

Figure 7: Tracing out an index

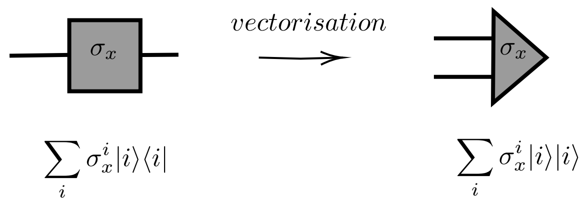


Figure 8: Vectorisation of an Operator

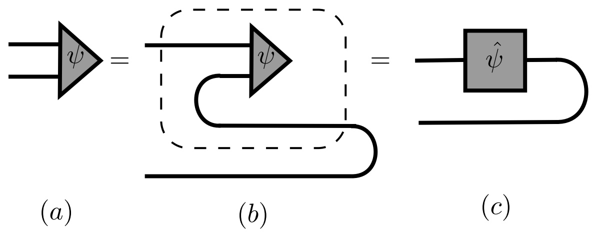


Figure 9: This is 2-party Quantum State $|\psi\rangle$. It can also be interpreted as an operator $\hat{\psi}$ acting on the bell state. The state $|\psi\rangle$ and operator $\hat{\psi}$ is related by wire-bending.

1.1.1.1 Map - State Duality

1.1.1.2 Schmidt Decomposition

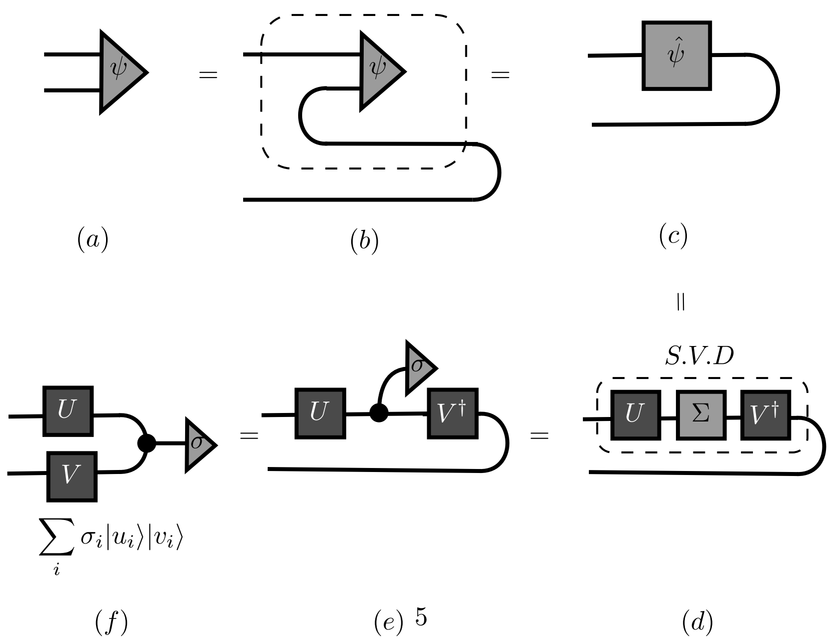
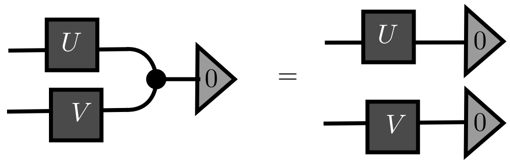


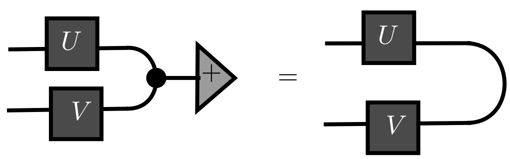
Figure 10: (f) is the Schmidt Decomposition of the 2 party State

The vector $|\sigma\rangle$ provides information about the amount of entanglement between the two subsystems of $|\psi\rangle$.



(g)

Figure 11: (g) represent Schmidt Decomposition of Separable State



(h)

Figure 12: (h) represents Schmidt Decomposition of a maximally Entangled State

If $|\sigma\rangle$ has only one non-zero component $|\psi\rangle$ will be a separable state and will have no entanglement whereas if all the components of $|\sigma\rangle$ are non-zero and equal then $|\psi\rangle$ corresponds to maximally entangled state.
The components of $|\sigma\rangle$ are called the Schmidt coefficients and the quantity $E = -\sum_i \sigma_i log(\sigma_i)$ is called the *entanglement entropy* and gives a quantitative measure of entanglement between bi-partitions.

2 Scope and Objective

2.1 Scope

Quantum Error Correction(QEC) is inevitable if we want to realise scalable Quantum Computers.Moreover the theory of Quantum Error Correction is surprisingly linked to various different parts of theoretical physics like Holography[14] and Topological matter.

2.2 Objectives

- To study the formalism of Quantum Channels.
- To study the formalism of Quantum Error Correction.
- To study graphical tensor networks and apply them to Quantum Information.
- To make simulations of QEC protocols in Qiskit and deploy them on IBMQ.
- To Study application of Tensor Networks in Quantum Machine Learning.

3 Open Quantum Systems

Open Quantum Systems are system which interacts with the environment.All real world systems are open.Sufficiently isolated systems can be treated as closed.Isolated systems evolve according to the Schrödinger equation while Open Systems evolve according to the Lindblad Equation.Quantum channels are used to model open system dynamics with some constraints.

3.1 Quantum Channels

Time evolution of an isolated system is represented by a Unitary Operator ($\hat{U}$) acting on the systems wave function. This is done so that probabilities are conserved.

$$\langle\psi(t)|\psi(t)\rangle = \langle\psi(0)|\hat{U}^{\dagger}\hat{U}|\psi(0)\rangle = \langle\psi(0)|\psi(0)\rangle \implies \hat{U}^{\dagger}\hat{U} = I$$

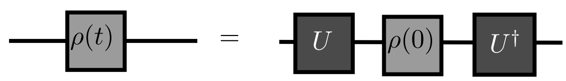


Figure 13: A Unitary transformation acting on a density matrix

In general, if the system is not fully isolated from the environment, the evolution will be global unitary on the system + environment.

3.1.1 Stinespring Representation

If $\rho_s(0) \in \mathscr{H}_s$ is the initial density matrix of the system and $\rho_e(0) = |e\rangle\langle e| \in \mathscr{H}_e$ the initial density matrix of the environment. The composite system-environment density matrix is given by:

$$\rho(0) = \rho_s(0) \otimes |e\rangle\langle e| \tag{1}$$

. The Quantum Channel ($\mathcal{E}$) performs a unitary operation U on the composite system. This generally entangles the initially separable states. The composite density matrix evolves into:

$$\rho(t) = U\rho(0)U^{\dagger} \tag{2}$$

$$\rho(t) = U(\rho_s(0) \otimes |e\rangle\langle e|)U^{\dagger} \tag{3}$$

Since the environment has a large number of degrees of freedom, it is almost impossible to keep track of all of them. We can get the reduced density matrix of the system by taking a partial trace over the environment. This is equivalent to discarding the environment degrees of freedoms.

$$\rho_s(t) = \mathcal{E}(\rho_s(0)) = tr_e[U(\rho_s(0) \otimes |e\rangle\langle e|)U^{\dagger}] \tag{4}$$

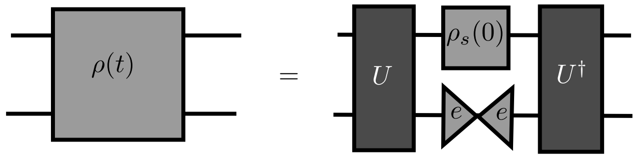


Figure 14: Evolution of system + environment density matrix under a global unitary

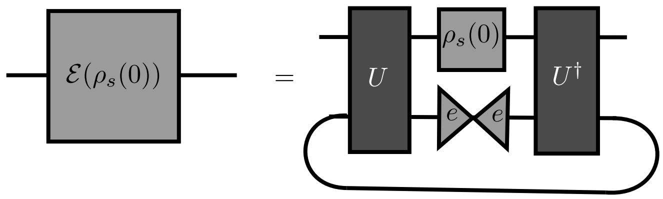


Figure 15: Tracing out the environments degrees of freedom

3.1.2 Kraus Representation

By expanding the initial environment density matrix on some arbitrary basis $|i\rangle$, we obtain the Kraus-Sudarshan representation of the quantum channel.

$$\rho_s(t) = tr_e[U(\rho_s(0) \otimes (\sum_i \lambda_i |i\rangle \langle i|))U^\dagger] \tag{5}$$

$$\rho_s(t) = \sum_j \langle j| (U(\rho_s(0) \otimes (\sum_i \lambda_i |i\rangle \langle i|))U^\dagger) |j\rangle \tag{6}$$

$$\rho_s(t) = \sum_{j,i} (\sqrt{\lambda_i} \langle j| U |i\rangle)\rho_s(0)(\lambda_i \langle i| U^\dagger |j\rangle) \tag{7}$$

$$\rho_s(t) = \sum_{j,i} K_{ij}\rho_s(0)K_{ij}^\dagger \tag{8}$$

Here K_{ij} is the Kraus-Sudarshan operator.
For the Kraus operators to map density matrices to density matrices there is an added constraint that it should preserve the trace.

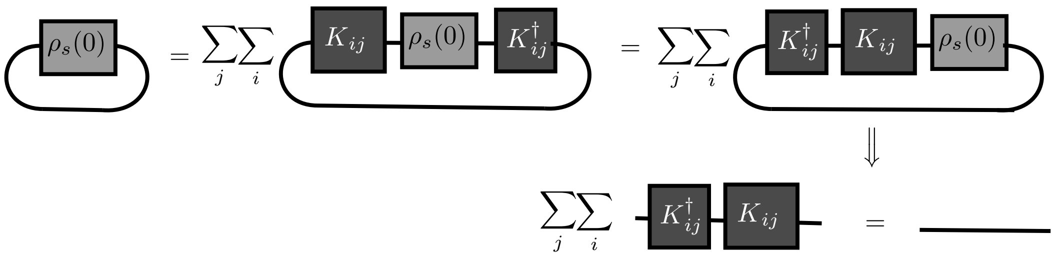


Figure 16: Kraus operators must be trace preserving

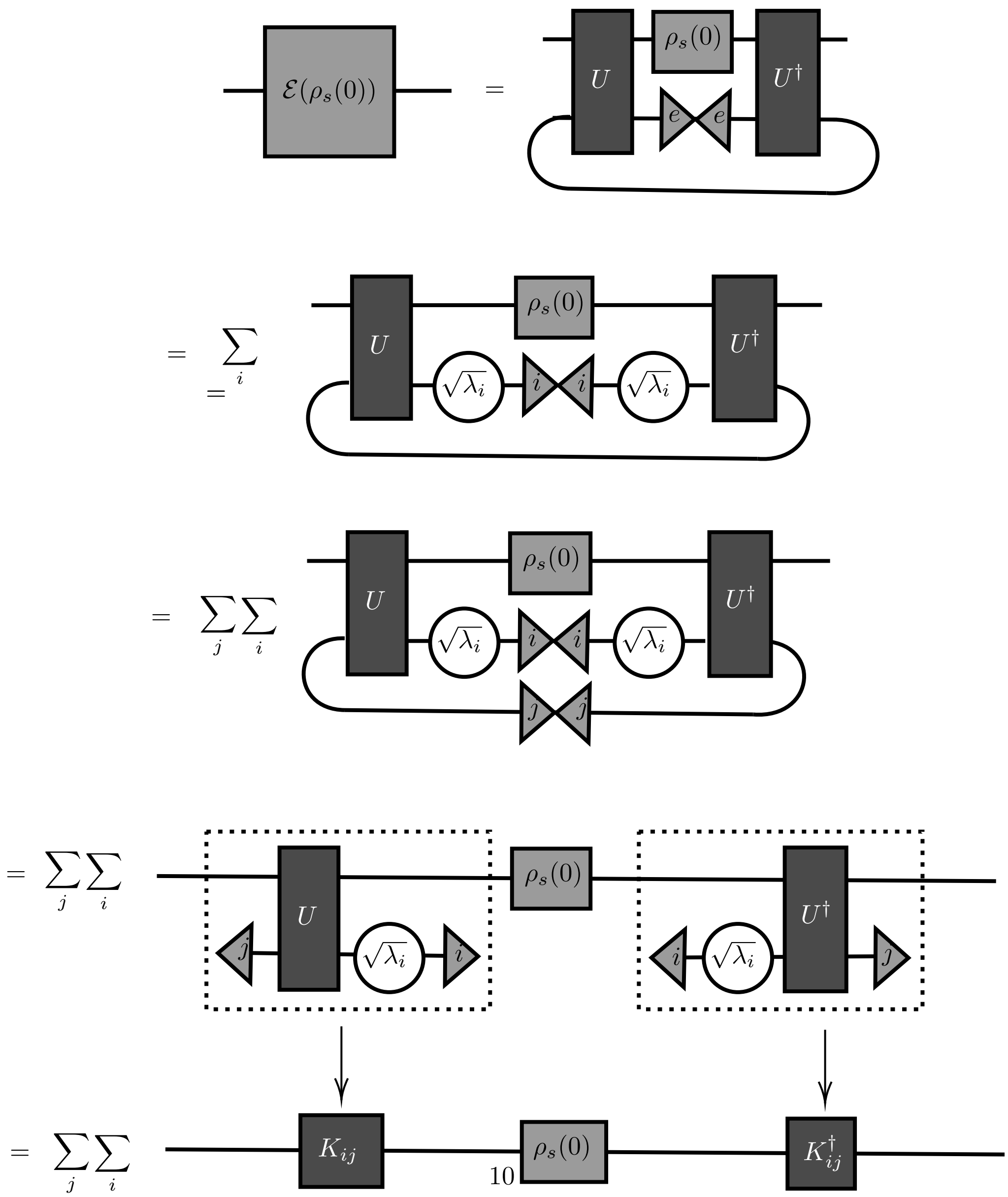


Figure 17: Diagrammatic conversion from Stinespring representation to Kraus representation

3.1.3 Superoperator Representation

This representation is based on the fact that the set of Pauli operators $(I, \sigma_x, \sigma_y, \sigma_z)$ forms a basis for hermitian operators and so the density matrix of the system can be decomposed as their linear combinations. Qubit is a quantum system of dimension N= 2. Let $[|0\rangle, |1\rangle]$ be an orthonormal basis. Pure qubit states can be expressed as linear combinations of these two vectors, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. Because of the normalisation condition $|\alpha|^2 + |\beta|^2 = 1$, the state can be written as:

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}\langle 1| \tag{9}$$

With parameters $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$ uniquely define a point $\tau = (x, y, z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ on a unit sphere, which is known as the Bloch sphere.

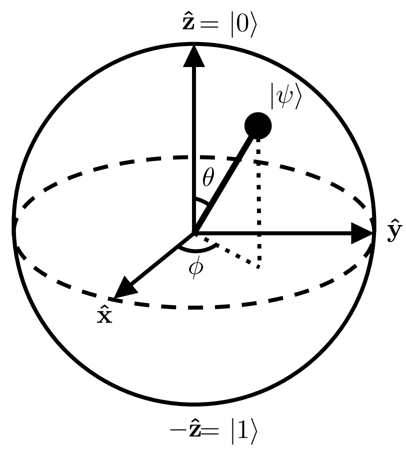


Figure 18: The Bloch ball

A density matrix that corresponds to a pure state $|\psi\rangle$ is:

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & -1+z \end{pmatrix} = \frac{1}{2}(I + \tau.\sigma) \tag{10}$$

Where σ is a vector of Pauli matrices, $\sigma = (\sigma_x, \sigma_y, \sigma_z), \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Mixed states are convex combinations of pure states. For every mixed qubit state ρ there exist states $|\psi_1\rangle, |\psi_2\rangle$ and $p \in \mathbb{R}, p \leq 1$, so that ρ can be expressed as:

$$\rho = p|\psi_1\rangle\langle\psi_1| + (1-p)|\psi_2\rangle\langle\psi_2| = \frac{1}{2}[I + (p\tau_1 + (1-p)\tau_2).\sigma] \tag{11}$$

Mixed qubit state can be represented with vector τ that lies inside the unit ball. This unit ball is called the Bloch ball, i.e., the space bounded by the Bloch sphere. Every qubit state corresponds to a point in the Bloch ball. If the state is pure, the point is on the sphere, $|\tau| = 1$ otherwise it lies inside,$|\tau| \leq 1$. Let $d = \frac{1}{2}(w, \tau) = 1/2 = \frac{1}{2}(w, x, y, z)$ so that $\rho = \mathbf{d}.(I, \sigma) = \mathbf{d}.(I, \sigma_x, \sigma_y, \sigma_z)$. The condition for ρ to be a valid density matrix is w = 1, $|\tau| \leq 1$ and the elements of d should be real. The action of a Quantum Channel on ρ can be viewed as a linear transformation from $\rho \to \rho'$ which can can be viewed as a linear transformation on d , i.e $[\mathbf{d} = \frac{1}{2}(1, x, y, x)] \to [\mathbf{d'} = \frac{1}{2}(1, x', y', x')], \mathbf{d'} = \mathbf{S}.\mathbf{d}$. Further the density matrix can be vectorised as illustrated in figure 8. Then the Superoperator can be represented graphically as a two index tensor.

Because of the condition that $\mathbf{S}$ should preserve the trace, $\mathbf{S}$ can be written as a block matrix.

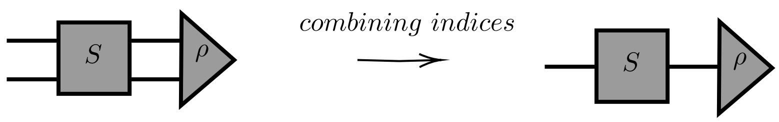


Figure 19: Combining indices of a Superoperator

$$\mathbf{S} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{t} & \mathbf{T} \end{pmatrix} \tag{12}$$

Here $\mathbf{0} = (0, 0, 0)$, $\mathbf{t}$ is a column matrix and $\mathbf{T}$ is a 3×3 matrix. The transformation $\tau \to \tau'$ becomes an affine transformation, $\tau' = \mathbf{T}.\tau + \mathbf{t}$. It is possible to write $\mathbf{T}$ as a product of two rotational matrices $\mathbf{O_1}, \mathbf{O_2}$ and a diagonal matrix $\mathbf{\Lambda}, \mathbf{T} = \mathbf{O_1 \Lambda O_1}$ by Singular Value Decomposition (SVD). Since rotational matrices only rotate the coordinate system, we restrict ourselves to the case $\mathbf{T} = \mathbf{\Lambda}$. Then, $\mathbf{t} = (t_x, t_y, t_z)$ and $\lambda = (\lambda_x, \lambda_y, \lambda_z)$ that determine, up to rotations, any arbitrary qubit channel. The transformation becomes:

$$\tau' = \begin{pmatrix} \lambda_x & 0 & 0 \\ 0 & \lambda_y & 0 \\ 0 & 0 & \lambda_x \end{pmatrix} \tau + \mathbf{t} = (\lambda_x t_x, \lambda_y t_y, \lambda_z t_z) \tag{13}$$

since $|\tau| \leq 1, \tau' = (x', y', z')$ lies inside the ellipsoid defined by,

$$(\frac{x' - t_x}{\lambda_x})^2 + (\frac{y' - t_y}{\lambda_y})^2 + (\frac{z' - t_z}{\lambda_z})^2 \leq 1 \tag{14}$$

Parameters $\lambda_x, \lambda_y, \lambda_z$ define scaling of the Bloch ball and parameters t_x, t_y, t_z define translation of the origin of the Bloch ball.

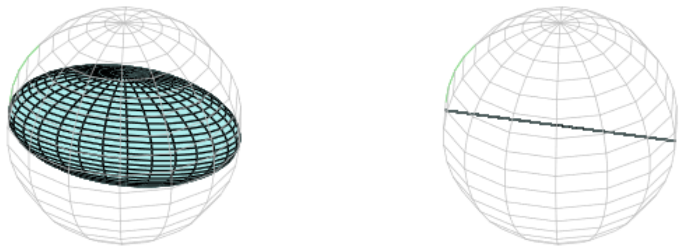


Figure 20: Evolution of Bloch ball under bit-flip map with two different bit - flip probabilities.Taken from [15]

The properties of **S** to be a valid superoperator can be expressed graphically as,

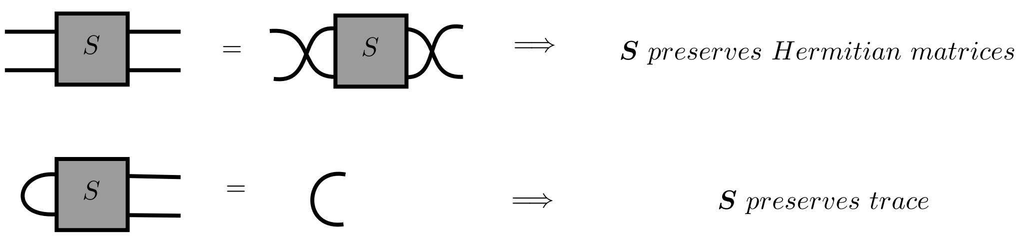


Figure 21: Properties of a Superoperator

3.1.4 Choi-Matrix Representation

This is a vectorisation of the Kraus operators. The Choi matrix of a Channel $\mathcal{E}$ represented by the set of Kraus operator K_i is given by,

$$C_{\mathcal{E}} = (I \otimes \mathcal{E}) |\Phi^+\rangle \langle\Phi^+| = (I \otimes \sum_i K_i) |\Phi^+\rangle \langle\Phi^+| (I \otimes \sum_i K_i^\dagger) \tag{15}$$

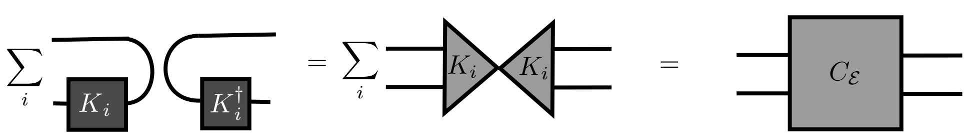


Figure 22: Choi matrix as a vectorisation of Kraus Operators

Where $|\Phi^+\rangle$ is the bell state. The evolution of the state in terms of Choi - Matrix is given by,

$$\mathcal{E}(\rho) = Tr_{\mathcal{X}}[(\rho^T \otimes I)C_{\mathcal{E}}] \tag{16}$$

$\mathcal{X}$ is the degree of freedom of the system. Equivalence between Choi - matrix representation and Kraus representation is shown in diagrammatic form.

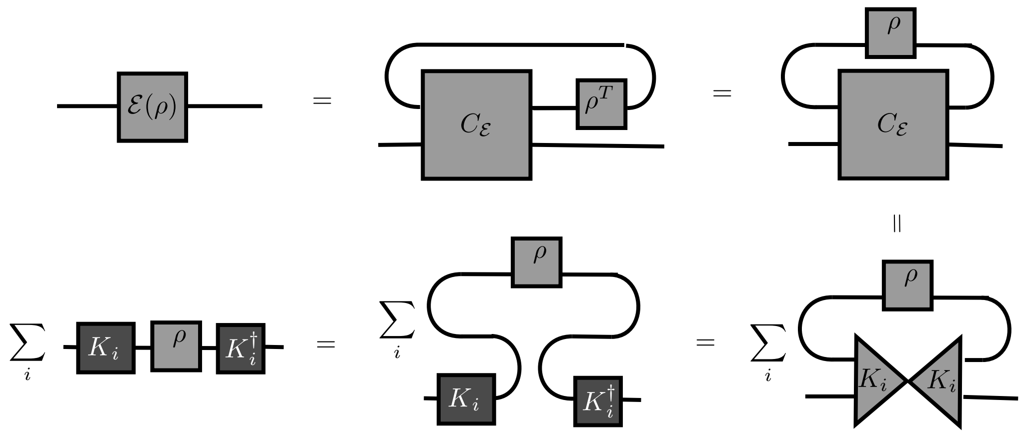


Figure 23: Equivalence between Choi - matrix representation and Kraus representation.

4 Quantum Error Correction

We have learned that it is possible to fight entanglement with entanglement.

– John Preskill

A quantum computer operates on a quantum system (S) by applying a series of gates on S and making a measurement in an appropriate basis. But, one of the main practical problems is presented by decoherence, which is the phenomenon by which S interacts and gets entangled with the environment in an uncontrolled way, such that, the final state of the system is not the one that we would expect if S was isolated. An ideal gate will map states living on the surface of the Bloch ball to its surface but, decoherence can map states living on the surface to the inside of the Bloch ball.

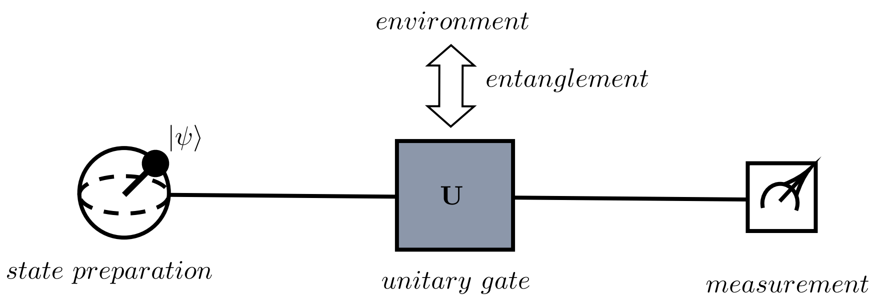


Figure 24: Illustration of a noisy quantum operation

Such noisy operations can be modelled by a completely positive trace-preserving (CPTP) linear map or quantum channel($\mathcal{N}$) with Kraus operators (E_i).

$$\mathcal{N}(\rho) = \sum_i E_i \rho E_i^\dagger, \sum_i E_i^\dagger E_i = I \tag{17}$$

Quantum Error Correction is the procedure to find and revert such undesirable noises from our computations.

4.1 Quantum Error Correcting Codes(QECC)

Given a quantum system with Hilbert space $\mathscr{H}_S$, a quantum error correcting code (QECC) is a subspace $C_S \subseteq \mathscr{H}_S$. It is also called the code subspace and states in C_S is called the code words or code states.

Given another quantum system with Hilbert space $\mathscr{H}_{S'}, dim\mathscr{H}_{S'} \leq dim\mathscr{H}_S$, an encoder is a quantum channel $\mathcal{E} : B(\mathscr{H}_{S'}) \rightarrow B(\mathscr{H}_S), \mathcal{E}(\rho_S) = W\rho_S W^\dagger \; \forall \; \rho_S \in B(\mathscr{H}_{S'})$, where $W : \mathscr{H}_{S'} \rightarrow \mathscr{H}_S$ is an

isometry ($W^\dagger W = I_{\mathscr{H}_S}$). The QECC will be the image of this encoder.
A QECC is exact w.r.t to a noise channel ($\mathcal{N}$) if there exists a Recovery Map ($\mathcal{R}$) such that,

$$\mathcal{R} \circ \mathcal{N} \circ \mathcal{E} = I_{\mathscr{H}_{S'}} \tag{18}$$

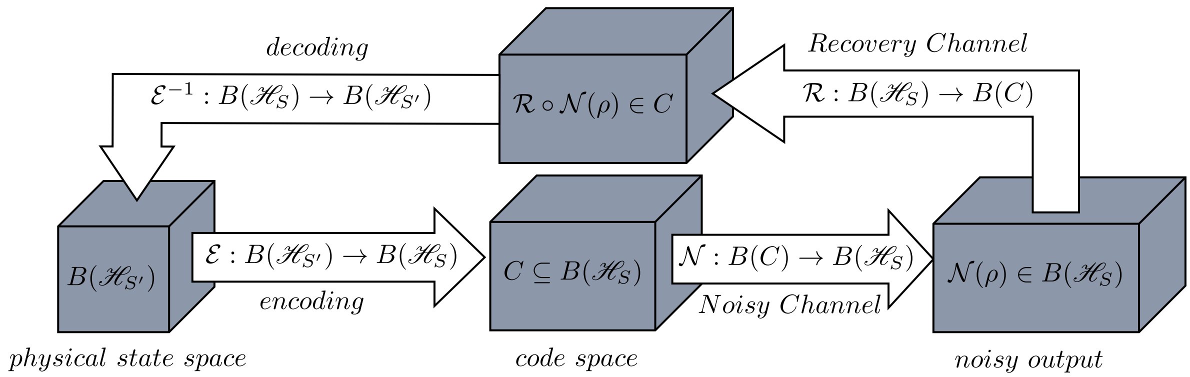


Figure 25: Illustration of an error correction scheme

4.2 Knill-Laflamme conditions

Given a noise channel $\mathcal{N}(\rho) = \sum_i E_i \rho E_i^\dagger$ and a QECC (C), a necessary and sufficient condition for the existence of a recovery map $\mathcal{R}$ correcting against $\mathcal{N}$ are given by : $PE_i^\dagger E_j P = \lambda_{ij} P$ where P is the projector onto C and $\lambda = (\lambda_{ij})$ is a density. By picking a basis$|a\rangle$ of C, it can be observed that $\langle a| E_i^\dagger E_j |b\rangle = \lambda_{ij}\delta_{ab}$, which means that orthogonal codewords remain orthogonal under the action of the noise.

PROOF:
Assume that there exists a recovery map $\mathcal{R}$ with Kraus operators R_j and Stinespring dilation $U_\mathcal{R} : |\psi\rangle \otimes |0\rangle_A \to \sum_j R_j |\psi\rangle \otimes |A\rangle_A$, where A is some ancilla system. Then for $\mathcal{R}$ to correct $\mathcal{N}$:

$$U_\mathcal{R} U_\mathcal{N} : |\psi\rangle \otimes |0\rangle_E \otimes |0\rangle_A \to \sum_{ij} R_j E_i |\psi\rangle \otimes |i\rangle_E \otimes |j\rangle_A = \sum_{ij} \tilde{\lambda}_{ij} |\psi\rangle \otimes |i\rangle_E \otimes |j\rangle_A = |\psi\rangle \otimes |junk\rangle_{EA} \, for \, |\psi\rangle \in C \tag{19}$$

We then have that:

$$PE_i^\dagger E_j P = PE_i^\dagger (\sum_k R_k^\dagger R_k) E_j P = P(\sum_k \tilde{\lambda}_{ik}^* \tilde{\lambda}_{kj}) P = \lambda_{ij} P \tag{20}$$

Where λ_{ij} is Hermitian and $\langle u| E_i^\dagger E_j |v\rangle = \lambda_{ij}\delta_{uv}$ and $|u\rangle , |v\rangle$ orthogonal vectors in codespace.

Since λ_{ij} is a Hermitian matrix it can be diagonalized using a unitary matrix, i.e.,

$$\lambda_{ij} = \sum_k u_{ik} d_k u^*_{jk} \tag{21}$$

With eigenvalues $d_k \geq 0$

With this, we can now define a new set of operators $F_k = \sum_i u_{ik} E_i / \sqrt{d_k}$ when $d_k \neq 0$.These errors are called principle errors.When $d_k =, F_k = \sum_i u_{ik} E_i$,these are called null errors. The new opertators F_k spans the set of possible Errors $\{E_i\}$.Now the Knill - Laflamme conditions becomes,

$$PF_k^\dagger F_l P = \delta_{kl} P \implies \langle u| F_k^\dagger F_l |v\rangle = \delta_{kl}\delta_{uv}. \text{ when } d_k \neq 0$$
$$\langle u| F_k^\dagger F_k |v\rangle = 0 \text{ when } d_k = 0$$

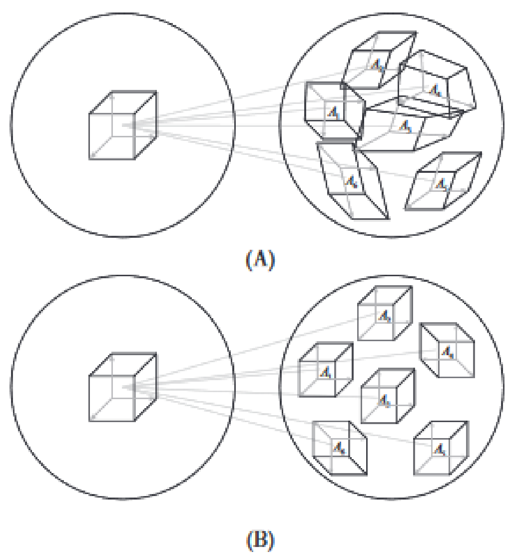


Figure 26: (A): Bad code with non-orthogonal overlapping resultant spaces (B): Good code with orthogonal distinguishable resultant spaces.(figure taken from Nielsen & Chuang 2010)

Physically this means that by using some ancilla we move the entanglement between system and environment, induced by the noise channel, to entanglement between environment and ancilla. An error correction scheme extracts entanglement entropy from the system and puts it into the ancilla. If we then want to reuse the same ancilla for a second round of error correction, we have to first reinitialise it, which costs energy. This is reminiscent of a sort of refrigeration.

4.3 Stabilizer Formalism

An important class of quantum error correcting codes is given by stabilizer codes.

The Pauli group: $\mathcal{G}_n$ of n qubits is defined as the group with elements given by all the possible n-fold tensor products of Pauli matrices, where each tensor-product factor can be independently an I,X,Y or Z, together with multiplicative factors $\pm 1, \pm i$, where the group operation is the matrix multiplication. Those elements with a multiplicative factor +1 are called Pauli operators. Examples:

$$\mathcal{G}_2 : \{\pm I \otimes I, \pm iI \otimes I, \pm X \otimes I, \pm iX \otimes I, \pm Y \otimes I, \pm iY \otimes I, \pm Z \otimes I, \pm iZ \otimes I, \pm I \otimes X, \pm iI \otimes X, \pm X \otimes X, \pm iX \otimes X, \pm Y \otimes X, \pm iY \otimes X, \pm Z \otimes X, \pm iZ \otimes X, \pm I \otimes Y, \pm iI \otimes Y, \pm X \otimes Y, \pm iX \otimes Y, \pm Y \otimes Y, \pm iY \otimes Y, \pm Z \otimes Y, \pm iZ \otimes Y, \pm I \otimes Z, \pm iI \otimes Z, \pm X \otimes Z, \pm iX, \pm Y \otimes Z, \pm iY \otimes Z, \pm Z \otimes Z, \pm iZ \otimes Z\}$$

Stabilizer group ($\mathcal{S}_n$): It is an abelian subgroup of the Pauli group $\mathcal{G}_n$ that does not contain $-I^{\otimes}$. A stabilizer group can be compactly specified by $m \leq n$ independent generators $S_1, ..., S_m$(i.e. the minimal number of elements such that any other one can be expressed as a product of them) and we use the notation $\mathcal{S}_n = \langle S_1, ..., S_m \rangle$ to mean that $\mathcal{S}_n$ is generated by the given set. All elements in $\mathcal{S}_n$ have eigenvalues in $\{1, -1\}$. This can be seen by noting that $S_i^2 = I^{\otimes n}$ for all $S_i \in \mathcal{S}_n$.

Error Syndrome: Given a Pauli operator P in $\mathcal{G}_n$, we define its (error) syndrome w.r.t. a given set of stabilizer generators $\{S_{i=1}^m\} \in \mathcal{S}_n$ as the vector $k = (k_1, ..., k_m)$, where k_i is 1 if P commutes with S_i i.e $[S_i, P] = 0$ and 1 if P anti-commutes with S_i i.e $\{S_i, P\} = 0$.A stabilizer anti - commutes with an error, if the error kicks the codewords out of the +1 eigenspace of that stabilizer.

Stabilizer Code: Let $\mathscr{H} = (\mathbb{C}^2)^{\otimes n}$ and $\mathcal{S}_n$ be a stabilizer group for n qubits. A stabilizer code is given by the simultaneous +1-eigenspace C of the elements in $\mathcal{S}_n$.The states in C are called codewords.It can be seen that a pauli error P acting on any $|\psi\rangle \in C$ will take it outside C and by measuring the error syndrome we can determine P. we can revert the error by applying P once again since $P^2 = I^{\otimes n}$.

If $\mathcal{S}_n$ has m independent generators, it encodes $n - m$ logical qubits. This is because each generator divides the physical hilbert space in two, a +1 and a -1 eigenspace. since the physical hilbert space is 2^n dimensional the logical hilbert space will be 2^{n-m} dimensional.

Centralizer $\mathcal{Z}(\mathcal{S}_n)$: It is the subgroup of the Pauli Group ($\mathcal{G}$) that commutes with all elements in $\mathcal{S}_n$. If an error in $\mathcal{Z}(\mathcal{S}_n)$ occurs we cannot detect it using syndrome measurements.

Let E_a, E_b be two possible Pauli Errors. We cannot distinguish between them if they both share the same error syndrome, k i.e, both E_a and E_b commutes/anti - commutes with the same set of stabilizers in $\mathcal{S}_n$. Then $[E_a^\dagger E_b, S] = 0 \ \forall S \in \mathcal{S}_n$ i.e, $E_a^\dagger E_b$ is an element in the centralizer $\mathcal{Z}(\mathcal{S}_n)$ if $E_a^\dagger E_b \in \mathcal{S}_n$ even though we cannot distinguish between them we can apply either E_a, E_b to revert the error since one of three possibilities can occur each of which trivially on the code subspace. $(E_a^\dagger E_b \in \mathcal{S}_n, E_a^2 = I^{\otimes n}, E_b^2 = I^{\otimes n})$. Codes having this property are called *degenerate codes*. A QECC cannot simultaneously correct errors (E_a, E_b) if $E_a^\dagger E_b \in \mathcal{Z}(\mathcal{S}_n) - \mathcal{S}_n$ for nondegenerate codes, the equivalance between the above condition and the knill - laflamme conditions can be readily seen.

Code distance: The weight of a Pauli operator is the number of non-identity operators in it. For example, the operator $X \otimes X \otimes Z \otimes Y \otimes I \otimes I$ has a weight of 4. The code distance is the minimum weight of a Pauli operator that maps codewords to a different codeword. In stabilizer codes, the distance will be the minimum weight of operators in $\mathcal{Z}(\mathcal{S}_n)$. A code with a distance d can correct errors with a maximum weight of $\frac{d-1}{2}$. A code with n physical qubits, k logical qubits and d distance is called a [n,k,d] code.

Digitisation of noise: Suppose C is a quantum code and $\mathcal{R}$ is the error-correction operation to recover from a noise process $\mathcal{N}$ with Kraus Operators $\{E_i\}$. Suppose $\mathcal{F}$ is a quantum operation with Kraus Operators $\{F_j\}$ which are linear combinations of the E_i,that is $F_j = \sum_i m_{ji} E_i$ for some matrix m_{ji} of complex numbers. Then the Recovery operation $\mathcal{R}$ also corrects for the effects of the noise process $\mathcal{F}$ on the code C.This can be seen as follows,

$$\mathcal{U}_\mathcal{R}\mathcal{U}_\mathcal{F}(|\psi\rangle \otimes |0\rangle_E \otimes |0\rangle_A) = \mathcal{U}_\mathcal{R}(\textstyle\sum_j F_j |\psi\rangle \otimes |j\rangle_E \otimes |0\rangle_A) = \mathcal{U}_\mathcal{R}(\sum_j \sum_i m_{ji} E_i |\psi\rangle \otimes |j\rangle_E \otimes |0\rangle_A) =$$

$$\textstyle\sum_j \sum_k \sum_i m_{ji} R_k E_i |\psi\rangle \otimes |j\rangle_E \otimes |k\rangle_A = \sum_{jki} m_{ji} \lambda_{ik} |\psi\rangle \otimes |j\rangle_E \otimes |k\rangle_A = |\psi\rangle \otimes |junk\rangle_{EA}$$

This can be understood as: whenever we measure the syndrome $|k\rangle_A$ the state collapses to one of the errors E_i corresponding to the observed syndrome.

Syndrome measurement:

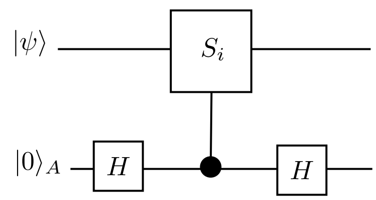


Figure 27: Circuit for measuring syndrome

The above circuit can be used to measure the syndrome k_i corresponding to the stabilizer S_i. Evaluation of the circuit yields,

$$\frac{1}{2}((I + S_i)\,|\psi\rangle) \otimes |0\rangle_A + \frac{1}{2}((I - S_i)\,|\psi\rangle) \otimes |1\rangle_A = (P_{+1})\,|\psi\rangle \otimes |0\rangle_A + (P_{-1})\,|\psi\rangle \otimes |1\rangle_A \qquad (22)$$

Here P_{+1} is the projector on to the $+1$ eigenspace of S_i and P_{-1} is the projector on to the -1 eigenspace of S_i. The state of the ancilla represents the syndrome.

4.3.1 Repetition codes

The qubit $|\psi\rangle$ is to be transmitted through a noisy channel which performs a bit-flip with a probability 'p'. If we transmit the qubit without encoding the data gets corrupted with probability 'p'.

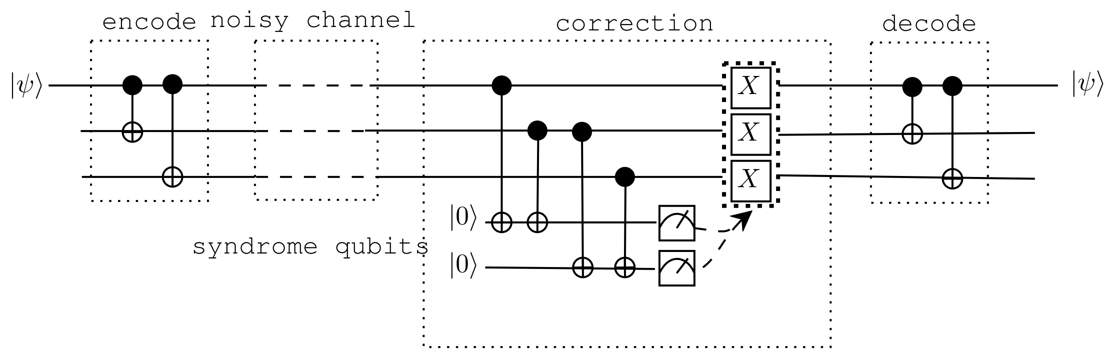


Figure 28: Circuit for repetition code

The state to be transmitted can be expanded in the computational basis as $\alpha|0\rangle + \beta|1\rangle$. After encoding, the combined state of three qubits becomes:

$$CNOT(q_0, q_1) \circ CNOT(q_0, q_2)[\alpha|000\rangle + \beta|100\rangle] = \alpha|000\rangle + \beta|111\rangle$$

Notation: $CNOT(q_1, q_2)|q_1\rangle|q_2\rangle = |q_1\rangle|q_1 \oplus q_2\rangle$

The encoding is an isometric map from a one dimensional Hilbert space $\mathscr{H}$ to a three dimensional Hilbert space $\mathscr{H} \otimes \mathscr{H} \otimes \mathscr{H}$. The valid codewords is a subspace of this Hilbert space spanned by $|000\rangle$ and $|111\rangle$.

Once this encoded state is sent through the noisy channel and if the probability of a bit flip occuring is 'p', one of 8 things can happen.

error	state	probability
E_0=I⊗I⊗I	$\alpha\|000\rangle + \beta\|111\rangle$	$(1\text{-p})^3$
E_1=X⊗I⊗I	$\alpha\|100\rangle + \beta\|011\rangle$	$\text{p}(1\text{-p})^2$
E_2=I⊗X⊗I	$\alpha\|010\rangle + \beta\|101\rangle$	$\text{p}(1\text{-p})^2$
E_3=I⊗I⊗X	$\alpha\|001\rangle + \beta\|110\rangle$	$\text{p}(1\text{-p})^2$
E_4=X⊗X⊗I	$\alpha\|110\rangle + \beta\|001\rangle$	$\text{p}^2(1-p)$
E_5=I⊗X⊗X	$\alpha\|011\rangle + \beta\|100\rangle$	$\text{p}^2(1-p)$
E_6=X⊗I⊗X	$\alpha\|101\rangle + \beta\|010\rangle$	$\text{p}^2(1-p)$
E_7=X⊗X⊗X	$\alpha\|111\rangle + \beta\|000\rangle$	p^3

After recieving the qubits, the reciever can detect the position of the error by measuring Z⊗Z⊗I and I⊗Z⊗Z.

Z$|0\rangle$ = $|0\rangle$, Z$|1\rangle$ = -1$|0\rangle$

stabilizer	state	eigenvalue
S_1=Z⊗Z⊗I	$\alpha\,|000\rangle + \beta\,|111\rangle$	1
S_2=I⊗Z⊗Z	$\alpha\,|000\rangle + \beta\,|111\rangle$	1
S_1=Z⊗Z⊗I	$\alpha\,|100\rangle + \beta\,|011\rangle$	-1
S_2=I⊗Z⊗Z	$\alpha\,|100\rangle + \beta\,|011\rangle$	1
S_1=Z⊗Z⊗I	$\alpha\,|010\rangle + \beta\,|101\rangle$	-1
S_2=I⊗Z⊗Z	$\alpha\,|010\rangle + \beta\,|101\rangle$	-1
S_1=Z⊗Z⊗I	$\alpha\,|110\rangle + \beta\,|001\rangle$	1
S_2=I⊗Z⊗Z	$\alpha\,|110\rangle + \beta\,|001\rangle$	-1
S_1=Z⊗Z⊗I	$\alpha\,|011\rangle + \beta\,|100\rangle$	-1
S_2=I⊗Z⊗Z	$\alpha\,|011\rangle + \beta\,|100\rangle$	1
S_1=Z⊗Z⊗I	$\alpha\,|101\rangle + \beta\,|010\rangle$	-1
S_2=I⊗Z⊗Z	$\alpha\,|101\rangle + \beta\,|010\rangle$	-1
S_1=Z⊗Z⊗I	$\alpha\,|111\rangle + \beta\,|000\rangle$	1
S_2=I⊗Z⊗Z	$\alpha\,|111\rangle + \beta\,|000\rangle$	1

If, only a maximum of single bit flip occurs we can identify the position of the bit flip by measuring the eigenvalues of S_1, S_2.
The group generated by S_1 and S_2 under composition is called the stabilizer of the code. All valid codewords will have eigenvalue 1 for all elements of this group.

One can measure eigenvalues of S_1, S_2 using two syndrome qubits.First perform two CNOT's from the first and second qubit to the first syndrome,followed by two CNOT's from the second and third qubit to the second syndrome.

state	probability
$(\alpha\|000\rangle + \beta\|111\rangle)\|00\rangle$	(1-p)^3
$(\alpha\|100\rangle) + \beta\|011\rangle\|10\rangle$	p(1-p)^2
$(\alpha\|010\rangle + \beta\|101\rangle)\|11\rangle$	p(1-p)^2
$(\alpha\|001\rangle + \beta\|110\rangle)\|01\rangle$	p(1-p)^2
$(\alpha\|110\rangle + \beta\|001\rangle)\|01\rangle$	p$^2(1-p)$
$(\alpha\|011\rangle + \beta\|100\rangle)\|10\rangle$	p$^2(1-p)$
$(\alpha\|101\rangle + \beta\|010\rangle)\|11\rangle$	p$^2(1-p)$
$(\alpha\|111\rangle + \beta\|000\rangle)\|00\rangle$	p^3

Then the reciever can perform the following actions to recover the encoded qubit.

syndrome	action
00	do nothing
10	apply X to first qubit
01	apply X to third qubit
11	apply X to second qubit

This scheme can correct errors (E_0, E_1, E_2, E_3) which can have a maximum of 1 bit flip. The success probability of this scheme is $(1-p)^3 + 3p^3$ and fails with a probability $3p^2(1-p) + p^3$. For low 'p' this can work very well.
We can get rid of the syndrome qubits by using the circuit below. The gate at the end is the Toffoli gate or CCX gate.

$$CCX(q_0, q_1, q_2)\,|q_0, q_1, q_2\rangle = |q_0 \oplus (q_1 \wedge q_2), q_1, q_2\rangle$$

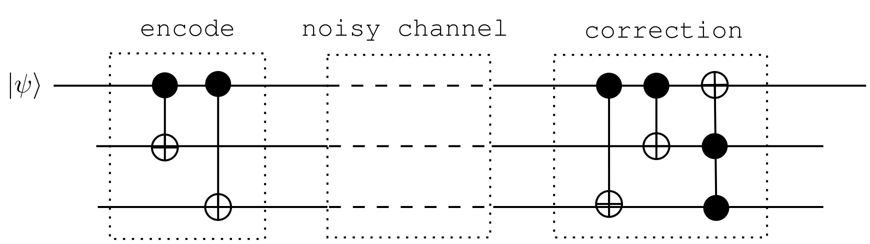


Figure 29: Circuit for Repetition Code without Syndrome

4.3.2 9-Qubit Shor Code

3-Qubit Repetition Code can only correct bit-flip errors or phase errors. For a code to be able to correct an arbitrary continuous error it is sufficient for it to be able to correct bit-flip (σ_x) and phase error (σ_z) together. Such a code can correct σ_y error also since $\sigma_y = i\sigma_x\sigma_z$. Any unitary error can be broken down into sum of these 3 errors. Shor 9-Qubit Code is an example of General Error Correcting Code. The encoding scheme is shown below,

$$|0\rangle \rightarrow |0\rangle_L = \frac{(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)}{2\sqrt{2}} \quad (23)$$

$$|1\rangle \rightarrow |1\rangle_L = \frac{(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)}{2\sqrt{2}} \quad (24)$$

	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8	Q_9
S_1	Z	Z	I	I	I	I	I	I	I
S_2	I	Z	Z	I	I	I	I	I	I
S_3	I	I	I	Z	Z	I	I	I	I
S_4	I	I	I	I	Z	Z	I	I	I
S_5	I	I	I	I	I	I	Z	Z	I
S_6	I	I	I	I	I	I	I	Z	Z
S_7	X	X	X	X	X	X	I	I	I
S_8	I	I	I	X	X	X	X	X	X

Stabilizer Group for the Shor 9-Qubit Code

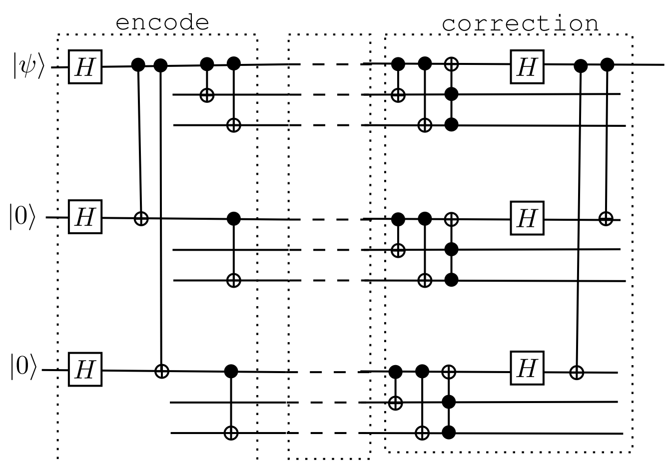


Figure 30: Circuit for 9-Qubit Shor Code

4.3.3 Toric Code

A qubit is placed on each link of a square lattice with periodic boundaries, i.e, the top edge is joined to the bottom edge and the left edge is joined to the right edge. This will yield a torus. This is the physical Hilbert space of the code. For a $N \times N$ Lattice there will be $2N^2$ edges (V-E+F = 2-2g ,g(no of genus =1, since a torus has a hole) and therefore, the Hilbert Space has dimension $n = 2^{2N^2}$.
For every square on the lattice (comprising 4 qubits, one on each link). A Stabilizer Generator A_p(called face Operator) is defined as:

$$A_p = Z \otimes Z \otimes Z \otimes Z \tag{25}$$

This is a Pauli Z operator acting on each of the four qubits and identity every where else.There are N^2 such Stabilizers one for each square in the Lattice.
Similarly, for each square on the dual Lattice , B_s(called Star Operator) is defined as:

$$B_s = X \otimes X \otimes X \otimes X \tag{26}$$

Each A_p, B_s has eigenvalues ± 1 (since, $A_p^2 = B_s^2 = I$). There will be N^2 such stabilizers one for each square in the Dual Lattice. All of these stabilizers mutually commute. It's easy to check for $[A_s, A_s] = [B_p, B_p] = 0$ because Pauli operators commute with themselves and I. More care is required with $[A_s, B_p] = 0$ since, these two terms either have 0 or 2 sites in common, and pairs of different Pauli operators commute, [XX,ZZ]=0. This is illustrated below:

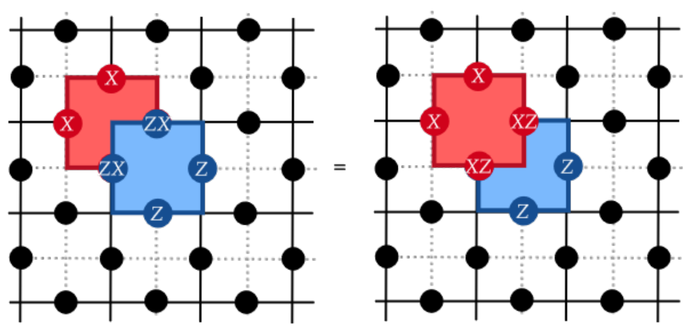


Figure 31: Neighbouring A_s and B_p commutes. The Dual Lattice is denoted by dotted lines.

The Stabilizer Group is the Group Spanned by A_s and B_p under Composition. This will be the set of all Loops of Z - operators on the real Lattice and Loops of X - operators on the dual Lattice with trivial Homology. This is illustrated below:

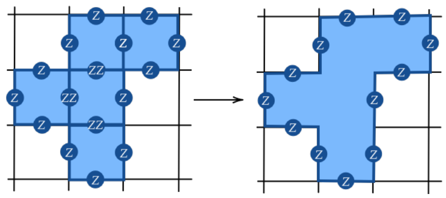


Figure 33

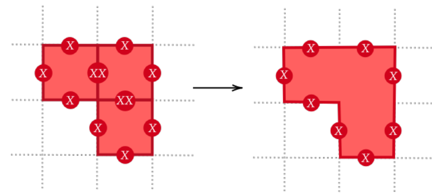


Figure 34

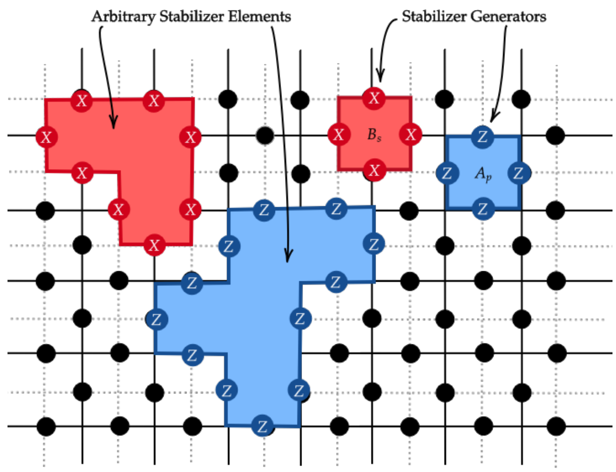


Figure 32: Stabilizer Group of the Toric Code

Since all A_p, B_s commute, the logical codespace is their simultaneous +1 eigenspace.

$\forall p : A_p |\psi\rangle = |\psi\rangle, \forall s : B_s |\psi\rangle = |\psi\rangle$ (27)

$\prod_p A_p = \prod_s B_s = I$ since each qubit is included in two stars and two faces. This means that one of the A_p and one of the B_s is dependent on all the others. Therefore, there will be $N^2 - 1$ independent A_p, B_s and the dimension (k) of the Stabilizer Group ($\mathcal{S}$) is $2N^2 - 2$. The dimension of the Logical Code Space will be $2^{n-k} = 2^{2N^2-(2N^2-2)} = 2^2$ and can encode 2 Qubits.
Loops with non-trivial homology (loops that winds around the Torus) will commute with the Stabilizer but will not lie in the Stabilizer Group. This is illustrated in Figure 36. This forms the set of

undetectable errors ($\mathcal{Z}(\mathcal{S}) - (\mathcal{S})$). The Logical Operators ($Z_{1,L}, Z_{2,L}, X_{1,L}, X_{2,L}$) are chosen to be

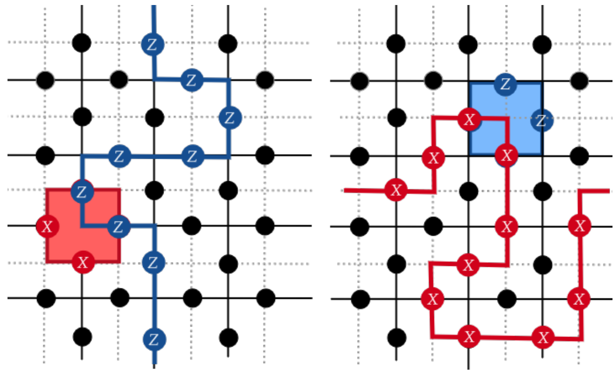


Figure 35: Loops of Z operators that winds around the Torus commutes with B_s since they meet at exactly 0 or 2 points. loops of X operators that winds around the torus commutes with A_p by the same logic.

the smallest weight independant operators in $\mathcal{Z}(\mathcal{S}) - (\mathcal{S})$ that generate the algebra of two qubits, i.e, commutation of operators on the two different logical qubits:

$$[X_{1,L}, X_{2,L}] = 0, [X_{1,L}, Z_{2,L}] = 0, [Z_{1,L}, Z_{2,L}] = 0, [Z_{1,L}, X_{2,L}] = 0 \tag{28}$$

and anti-commutation of the two on each qubit:

$$\{X_{1,L}, Z_{1,L}\} = 0, \{X_{2,L}, Z_{2,L}\} = 0 \tag{29}$$

The logical basis state is defined as:

$$|\psi_{x,y}\rangle : Z_{1,L} |\psi_{x,y}\rangle = (-1)^x |\psi_{x,y}\rangle, Z_{2,L} |\psi_{x,y}\rangle = (-1)^y |\psi_{x,y}\rangle \text{ for x,y} \in \{0,1\}$$

The weight of $X_{1,L}, Z_{1,L}, X_{2,L}, Z_{2,L}$ will be N hence, Toric Code has a distance N.
The observed eigenvalues of the stabilizers provide a “syndrome” that can be used to diagnose errors. If there are no errors in the code block, then every syndrome takes the value 1. Since each stabilizer is associated with a definite position on the surface, a site of the lattice or the dual lattice, the syndrome can be listed by all positions where the stabilizers take the value 1. It is convenient to regard each such position as the location of a particle, a “defect” in the code block. If errors occur on a particular chain (a set of links of the lattice or dual lattice), then defects occur at the sites on the boundary of the chain. These are called Anyon Chains and the defects are called Anyons. This error can be corrected by linking the Anyons to form a homologically trivial loop. Such loops will be present in the stabilizer and will act trivially on the codespace. For any given error chain,

there will be many different ways to connect the endpoints and thus the Toric Code is degenerate. Usually, the chain with the lowest weight is chosen.Finding the most optimal chain for correction is called as the syndrome decoding problem.It is in general an NP hard problem and one can use machine learning algorithms for doing this.
The Stabilizer Hamiltonian(H_s) is defined as:

$$H_s = -(\sum_p A_p + \sum_s B_s) \tag{30}$$

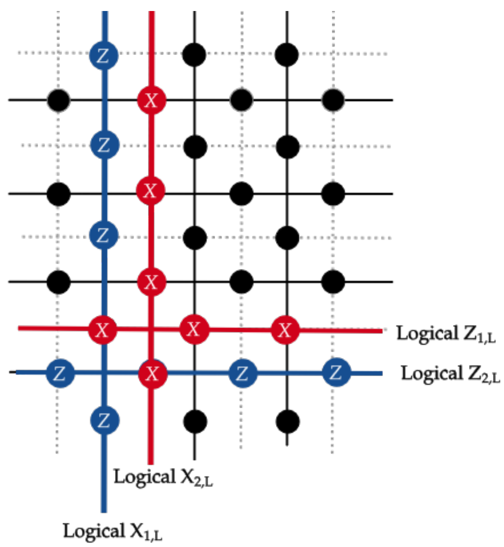


Figure 36: Operators that are supposed to commute do not meet and operators that are supposed to anti-commute meet at exactly one site, with an X and a Z

The lowest energy state of this Hamiltonian will be the state where there is no error and all syndromes take the value 1. Any error chain(Z_C or X_c) will increase the energy by 2 and can be considered as a creation operator for a Quasi-Particle (Anyons). Anyons are analogous to electric charges because they are always formed in pairs.

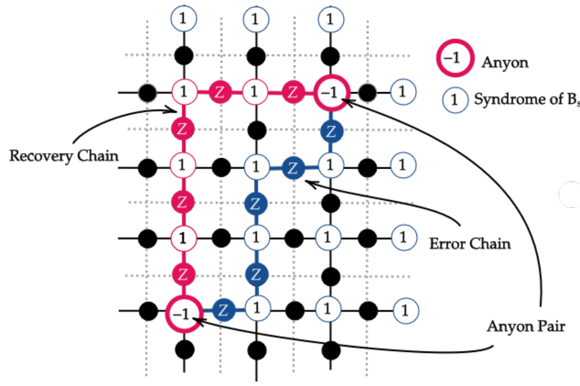


Figure 37: Correction of a chain of Z errors in the real lattice

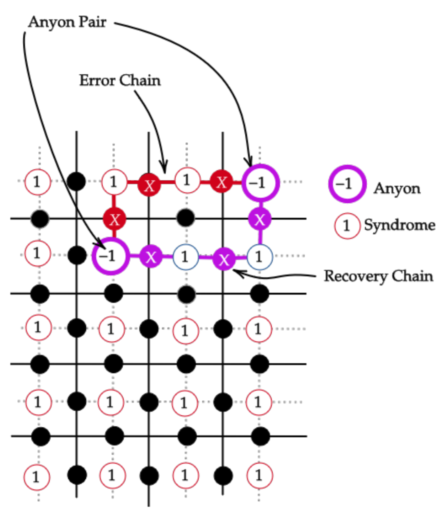


Figure 38: Correction of a chain of X errors in the dual lattice

4.3.4 Surface Codes

Toric code corrects errors based on their topological features. Such stabilizer codes are called Topological Codes. Surface Codes are a special class of topological codes, where the physical

Hilbert space of the code is a set of Qubits placed on a graph that can be embedded on a 2D surface.Toric code is a surface code where the underlying surface is a Torus.

4.3.4.1 Planar Surface Codes

Planar surface code is a surface code with the physical qubits embedded on a plane. These codes are easier to implement with the actual hardware.

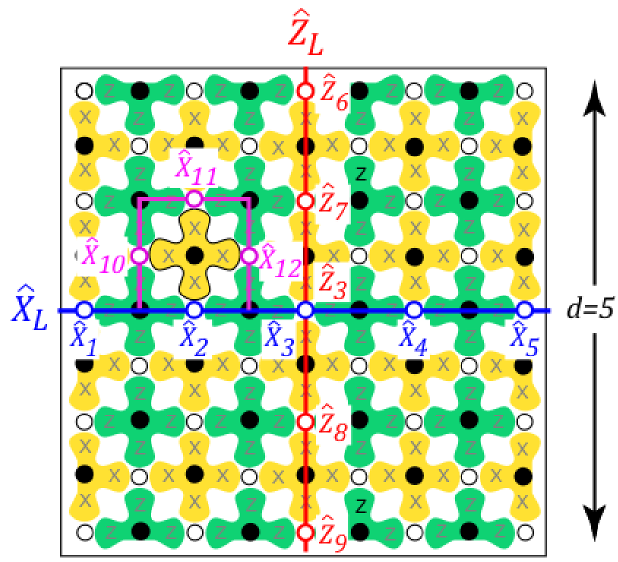


Figure 39: Planar Surface Codes, Source[16]

in the above figure the white dots are the data qubits and the black dots are the syndrome qubits used for correction.Here there are 41 data qubits(i.e n = 41) and 40 stabilizers(i.e k = 40). This code can store 1 logical Qubit(n-k).The chains($\hat{X_L}, \hat{Z_L}$) are the logical X and Z operators.These operators satisfy the usual commutation relations for X and Z operators. Since these operators have a weight 5 the distance of the code is 5.

We can introduce logical qubits on to a surface code by punching holes in the underlying graph.That is by turning off X or Z Stabilizer. If we turn of X stabilizers it is called an X-cut qubit other wise a Z-cut Qubit.

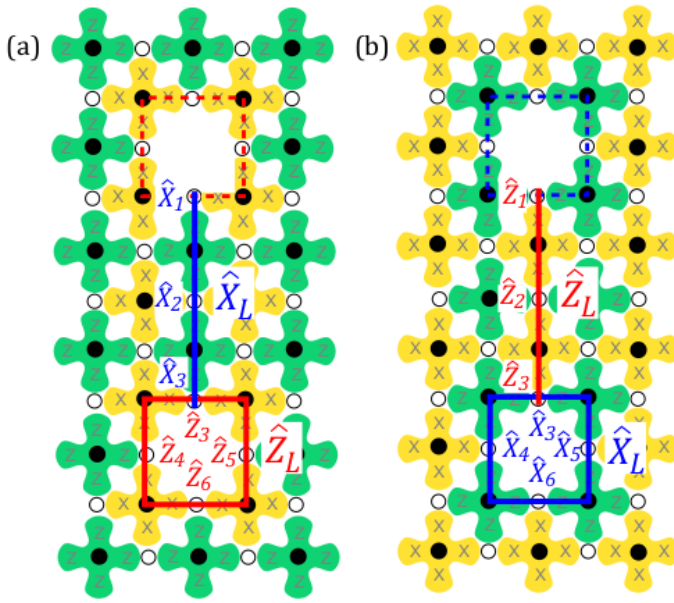


Figure 40: Z-cut and X-cut qubits, Source [16]

the logical operators(X_L and Z_L) satisfy the usual commutation relations. The distance of the code can be increased arbitrarily by increasing the weight of the logical operators.These defect qubits can be moved throughout the lattice by appropriately switching off and turning on stabilizers sequentially.This can be used to perform logical CNOT gates by braiding a Z-cut qubit around a X-cut qubit.

4.3.4.2 Logical CNOT by Braiding

there are different schemes for doing logical two qubit CNOT gates in a surface code architecture. Braiding defect qubits,Lattice Surgery,using twist defects. Here i am exploring the method of braiding defect qubits.

the CNOT gate have the following action on logical operators $\hat{X}, \hat{Y}$ in heisenberg picture.

$$CNOT^{\dagger}(\hat{I} \otimes \hat{X})CNOT = \hat{I} \otimes \hat{X}$$
$$CNOT^{\dagger}(\hat{X} \otimes \hat{I})CNOT = \hat{X} \otimes \hat{X}$$
$$CNOT^{\dagger}(\hat{I} \otimes \hat{Z})CNOT = \hat{Z} \otimes \hat{Z}$$
$$CNOT^{\dagger}(\hat{Z} \otimes \hat{I})CNOT = \hat{Z} \otimes \hat{I}$$

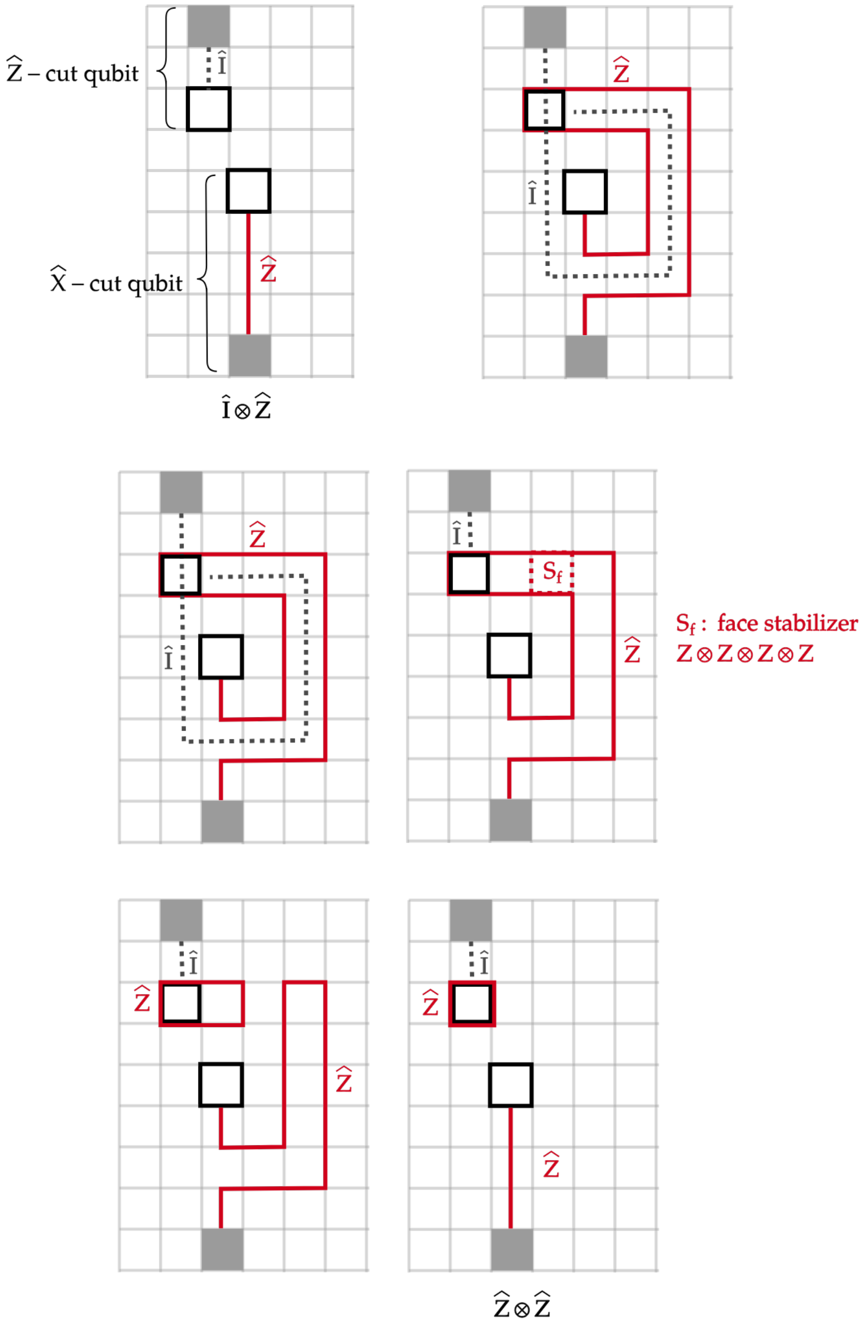


Figure 41: $\mathrm{CNOT}^{\dagger}(\hat{I} \otimes \hat{Z})CNOT = \hat{Z} \otimes \hat{Z}$

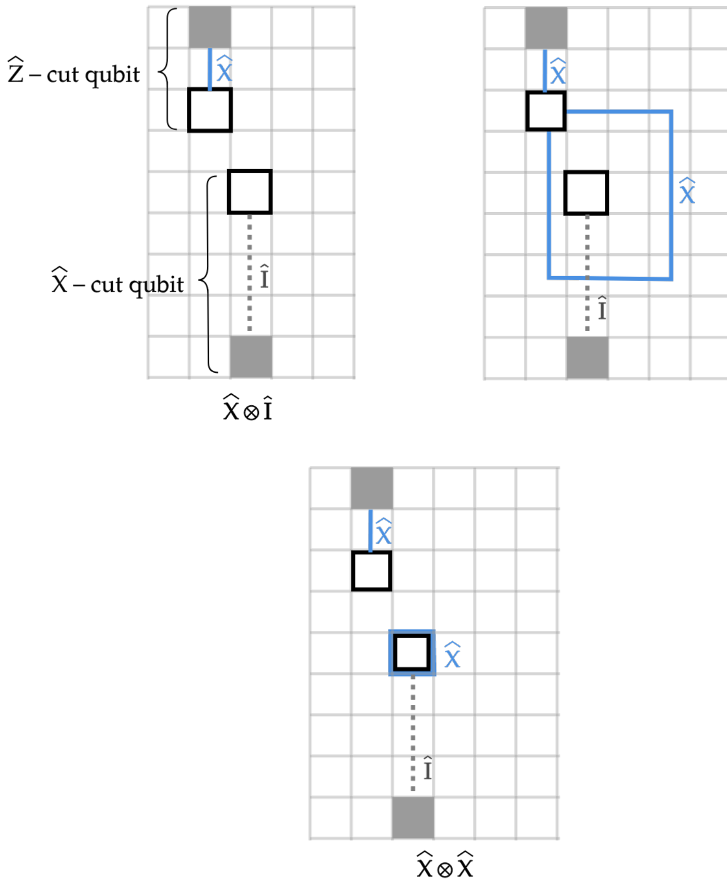


Figure 42: $\text{CNOT}^{\dagger}(\hat{X} \otimes \hat{I})CNOT = \hat{X} \otimes \hat{X}$

5 Hardware Bench-marking of IBMQ Quantum Computer

Quantum Computing is in the NISQ (Noisy Intermediate Scale Quantum computing) era. The Quantum Computers we have now are not ideal and Quantum Gates are often Noisy. Benchmarking is done to analyse the efficiency of a Quantum Computer. Here I am trying to Benchmark the performance of the IBMQ Quantum Computer which anyone can use online for doing experiments. The fidelity(f) between two states $|\psi_1\rangle$ and $|\psi_2\rangle$ is the inner product between the two. This is a measure of closeness between the two states. The programing was done using Qiskit[11].

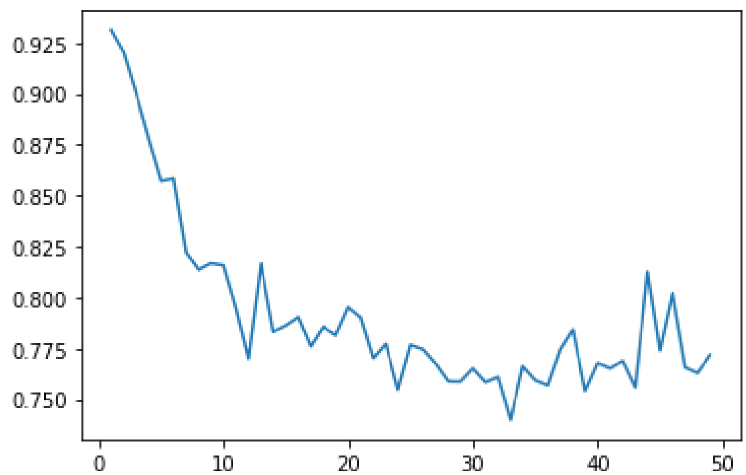


Figure 43: Plot of fidelity vs number of two qubit gates

6 Quantum Machine Learning

Quantum machine learning is an emerging field at the interface of quantum computing and machine learning. Both quantum circuits as well as neural networks (class of Machine Learning technique) can be modelled as tensor networks. Tree tensor networks are a type of tensor network used in condensed matter physics. Here, I am trying to make a binary classifier that can classify between two types of flowers based on 4 real-valued features (the length and the width of the sepals and petals) based on [10]. The classification is done by optimising a parametric tree tensor network based on a quantum circuit. The feature vectors are scaled element-wise to lie in $[0, \frac{\pi}{2}]$ and are encoded as the state of a 4-qubit wavefunction by the following encoding scheme:

$$|\psi_n^d\rangle = \cos(x_n^d)\,|0\rangle + \sin(x_n^d)\,|1\rangle \tag{31}$$

Here x_n^d is the n-th feature of the d-th feature vector. The four qubit input wavefunction after encoding becomes:

$$|\psi_1^d\rangle \otimes |\psi_2^d\rangle \otimes |\psi_3^d\rangle \otimes |\psi_4^d\rangle \tag{32}$$

This state is passed through the following parametric TTN circuit:

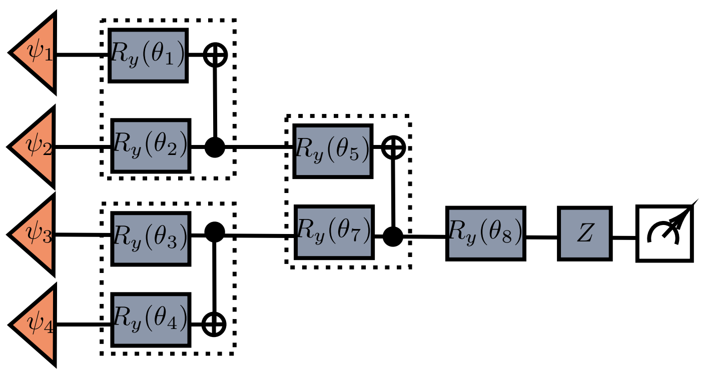


Figure 44: Parametric TTN Classifier

The expectation value of the measurement is considered as the predicted label for the flower. The parameters $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7, \theta_8$ are optimised using gradient descent for the given training set. The optimistion was done using PennyLane[11] and Qiskit[10].
The performance of the Circuit is given below.

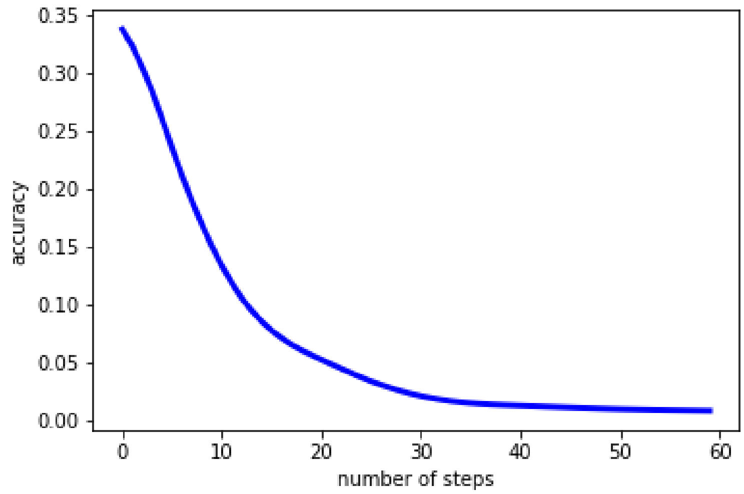


Figure 45: Plot of Cost vs Number of Steps

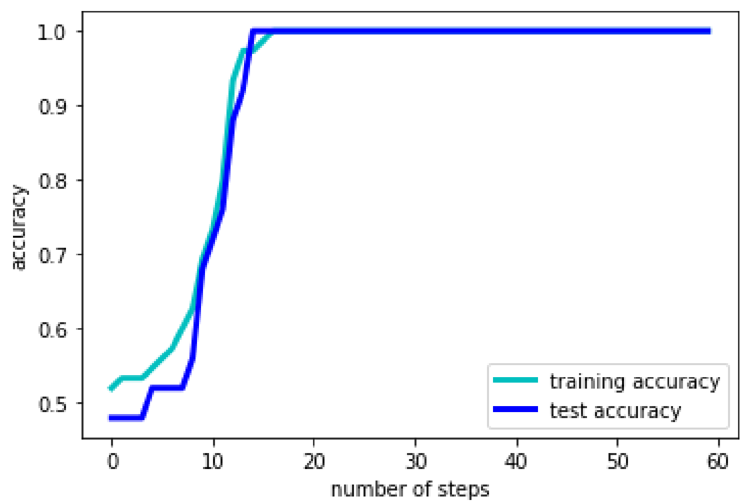


Figure 46: Plot of Training and Test accuracy vs Number of Steps

7 Summary

Quantum Computing has the potential to provide a paradigm shift in computing. To realise this potential, Error Correcting codes are crucial. Through this research work, an extensive study of Quantum Error Correction was carried out. Simulations were done for variational Quantum Error Correcting codes. The most promising codes are surface codes, therefore attempt was made to simulate arbitrary surface codes and experiment with decoding algorithms for the same.

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