# ELECTRODYNAMICS OF CAVITY RESONATORS

Project report (part-1)

Submitted in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE
in

PHYSICS

By

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**SURATHKAL, MANGALORE -575 025**
**DECEMBER 2014**
**PROJECT-I (PH 898)**

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## DECLARATION

I hereby declare that the report of P.G. Project Work entitled "**ELECTRODYNAMICS OF CAVITY RESONATORS**" which is being submitted to the National Institute of Technology Karnataka, Surathkal, in partial fulfillment of the requirements for the award of the Degree of Master of Science in the Department of Physics, is a bonafide report of the work carried out by me. The material contained in this report has not been submitted to any University or Institution for the award of any degree.

Place: NITK, Surathkal
Date:

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## CERTIFICATE

This is to certify that the P.G. project work entitled **ELECTRODYNAMICS OF CAVITY RESONATORS** submitted by **PRIYAMEDHA SHARMA K R**,(13451513PH10) as the record of work carried out by him is accepted as the P.G.project work report submission in partial fulfilment of the requirements for the award of the Degree of Master of Science in the Department of Physics.

**Chairman - DPGC**

**Dr. Deepak Vaid**
Project Adviser

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## ABSTRACT

Starting from the Maxwell's equations, many essential concepts of classical electrodynamics have been reviewed. Modal propagation of confined electromagnetic waves has been illustrated in the case of parallel plate wave guide and rectangular wave guide by using Maxwell's equations. The concepts and tools hence developed have been utilised in the analysis of rectangular resonating cavity. Optical cavities and micro cavities are introduced leading to the study of spontaneous emission in micro cavities.

# Contents

# Chapter 1

# INTRODUCTION

High frequency electromagnetic waves can be transferred from one place to another using hollow metallic structures called wave guides. The behaviour of electromagnetic waves is quite different when they are made to propagate through wave guides. They are no more transverse in general and they do exhibit different modes. When the open ends of these wave guides are closed by metal plates of high conductivity we have the so called cavity resonator. A cavity resonator is basically a closed structure which stores electromagnetic energy. The electric and magnetic fields are present in the cavities in the form of standing waves.

The main objective of this project work is to investigate spontaneous emission in optical micro cavities. As a part of the work, this report contains, in its first part, the study of essential elements of basic electrodynamics including Maxwell's equations, solutions of Maxwell's equation, gauge transformations. In the second part analysis of parallel plate wave guide and rectangular wave guide is carried out. A study of variation of impedance in rectangular wave guides with frequency has been done. Finally we have solved for rectangular resonating cavity. Then we turn our attention to optical cavities. Micro cavities and spontaneous emission in optical cavities have been introduced which are needed in further investigation of spontaneous emission in micro cavities in the course of our work.

# Chapter 2

# REVIEW OF ELECTRODYNAMICS

## 2.1 Review of Maxwell's equations

### 2.1.1 Maxwell's equations

The equations governing electromagnetic phenomena are known as Maxwell's equations. Before Maxwell, electrodynamics had four fundamental equations as follow.

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{2.1}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{2.2}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{2.3}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \tag{2.4}$$

Where $\vec{E}$ is the electric field and $\vec{B}$ is the magnetic field, $\rho$ is the charge density and $\vec{J}$ is the current density. Here, equation (2.1) is called Gauss's law in electrostatics, (2.3) is called Faraday's law and (2.4) is called Ampere's law in magnetostatics. If we apply divergence to the Ampere's law we get,

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 (\vec{\nabla} \cdot \vec{J}) \tag{2.5}$$

the left side must be zero, but the right side, in general, is not. The divergence of $\vec{J}$ fails to vanish in the case of non-steady currents. Maxwell corrected this discrepancy of Ampere's law to hold outside magnetostatics. Applying continuity equation and Gauss's law we have

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial (\epsilon_0 \vec{\nabla} \cdot \vec{E})}{\partial t} = -\vec{\nabla} \cdot \left( \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

hence by combining $\left(\epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$ with $\vec{J}$, in Ampere's law we get its general form:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Maxwell called his extra term the **displacement current**

$$\vec{J}_d = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Hence the **Maxwell's equations** are :

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{2.6}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{2.7}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{2.8}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{2.9}$$

Together with the force law,

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \tag{2.10}$$

They summerize the content of classical electrodynamics. The continuity equation

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \tag{2.11}$$

which is the mathematical expression of conservation of charge, can be derived from the Maxwell's equations by applying divergence to number (2.9).
In free space where $\rho$ and $\vec{J}$ vanish Maxwell's equations take the form:

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= 0 \\ \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \times \vec{B} &= \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \end{aligned}$$

[1, 2]

### 2.1.2 Maxwell's equations in matter

For the materials that are subject to electric and magnetic polarization the Maxwell's equations take a different form. For these materials there would be bound charges and currents. Hence Maxwell's equations are reformulated so as to make the explicit reference only to the free charges and currents.

In the static case, an electric polarization $\vec{P}$ produces a bound charge density

$$\rho_b = -\vec{\nabla} \cdot \vec{P}$$

Likewise a magnetic polarization $\vec{M}$ results in a bound current

$$\vec{J}_b = \vec{\nabla} \times \vec{M}$$

In the non-static case, any change in electric polarization involves a flow of bound charges which should be included in the total current.

$$\vec{J}_p = \frac{\partial \vec{P}}{\partial t}$$

Hence the total charge density can be separated into two parts:

$$\rho = \rho_f + \rho_b = \rho_f - \vec{\nabla} \cdot \vec{P}$$

and current density into three parts.

$$\vec{J} = \vec{J}_f + \vec{J}_b + \vec{J}_p = \vec{J}_f + \vec{\nabla} \times \vec{M} + \frac{\partial \vec{P}}{\partial t}$$

Gauss's law now becomes,

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0}(\rho_f - \vec{\nabla} \cdot \vec{P})$$

or

$$\vec{\nabla} \cdot \vec{D} = \rho_f$$

where, as in the static case

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

Meanwhile, Ampere's law (with Maxwell's term) becomes

$$\vec{\nabla} \times \vec{B} = \mu_0 \left( \vec{J}_f + \vec{\nabla} \times \vec{M} + \frac{\partial \vec{P}}{\partial t} \right) + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

or

$$\vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

where as before,

$$\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M}$$

Faraday's law and $\vec{\nabla} \cdot \vec{B} = 0$ are not affected by this separation of charge and current into free and bound parts.
Hence in terms of free charges and free currents Maxwell's equations become,

$$\vec{\nabla} \cdot \vec{D} = \rho_f \tag{2.12}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{2.13}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t} \tag{2.14}$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t} \tag{2.15}$$

For linear media,

$$\vec{P} = \epsilon_0 \chi_e \vec{E}, \quad and \quad \vec{M} = \chi_m \vec{H}$$

so

$$\vec{D} = \epsilon \vec{E} \quad and \quad \vec{H} = \frac{1}{\mu} \vec{B}$$

where $\epsilon = \epsilon_0(1 + \chi_e)$ and $\mu_0(1 + \chi_m)$. $\vec{D}$ is called electric displacement and,

$$\vec{J}_d = \frac{\partial \vec{D}}{\partial t}$$

### 2.1.3 Boundary conditions

In general, the fields $\vec{E}$,$\vec{B}$,$\vec{D}$,$\vec{H}$ will be discontinuous at a boundary between two different media, or at a surface that carries a charge density $\sigma$ or a current density $\vec{K}$.These can be deduced from the Maxwell's equations in integral form.

$$\oint_S \vec{D} \cdot d\vec{a} = Q_f \tag{2.16}$$

$$\oint_S \vec{B} \cdot d\vec{a} = 0 \tag{2.17}$$

$$\oint_P \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \left( \int_S \vec{B} \cdot d\vec{a} \right) \tag{2.18}$$

$$\oint_P \vec{H} \cdot d\vec{l} = I_f + \frac{d}{dt} \left( \int_S \vec{D} \cdot d\vec{a} \right) \tag{2.19}$$

If $\vec{D_1}$ and $\vec{D_2}$ are the electric displacements below and above an interface having a surface charge density $\sigma_f$ then equation (2.16) implies that the component of $\vec{D}$ that is perpendicular to the interface is discontinuous in the amount

$$D_1^{\perp} - D_2^{\perp} = \sigma_f \tag{2.20}$$

In the same way equation (2.17) implies that,

$$B_1^{\perp} - B_2^{\perp} = 0 \tag{2.21}$$

Equation (2.18) implies that the components of $\vec{E}$ parallel to the interface are continuous across the boundary.

$$\mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0 \tag{2.22}$$

Let $\vec{K}_f$ be a surface current on the interface and $\hat{n}$ a unit vector perpendicular to the surface then from equation (2.19) we can show that

$$\mathbf{H}_1^{\parallel} - \mathbf{H}_2^{\parallel} = \vec{K}_f \times \hat{n} \tag{2.23}$$

Hence the parallel components of $\vec{H}$ are discontinuous by an amount proportional to the surface current density. These are general boundary conditions for electrodynamics. [1, 2]

### 2.1.4 Electromagnetic waves

Consider Maxwell's equations in free space where there is no charge or current. They are a set of coupled, coupled first order partial differential equations for $\vec{E}$ and $\vec{B}$. Now consider,

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = \vec{\nabla} \times \left( -\frac{\partial \vec{B}}{\partial t} \right)$$

$$= -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

since $\vec{\nabla} \cdot \vec{E} = 0$, we have

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \tag{2.24}$$

Similarly,

$$\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \tag{2.25}$$

These two equations imply that electromagnetic waves travel through empty space with the velocity of light

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3.00 \times 10^8 \ m/s \tag{2.26}$$

The electromagnetic wave equations (2.24) and (2.25) have plane wave and spherical wave solutions. [1, 2]

### 2.1.5 Properties of electromagnetic waves

- Electromagnetic waves are transverse: the electric and magnetic fields are perpendicular to the direction of propagation. If we consider a plane wave travelling in the z-direction and $E_0$ and $B_0$ are the respective amplitudes then Faraday's law implies that $B_0 = \frac{k}{\omega}E_0 = \frac{1}{c}E_0$

- Monochromatic plane electromagnetic waves are polarized. The direction of $\vec{E}$ is used to specify the polarization of an electromagnetic wave.

- As the wave travels, it carries energy along with it. The energy flux density transported by the fields is give by the Poynting vector : $\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B})$. For monochromatic plane electromagnetic waves propagating int the z-direction, $\vec{S} = cu\hat{z}$ where u is the energy per unit volume in the electromagnetic fields and is given by $u = \epsilon_0 E^2$.

- The average power per unit area transported by an electromagnetic wave is called the intensity : $I = \frac{1}{2}c\epsilon_0 E_0^2$ [1]

## 2.2 Laplace's equation

In electrostatics[2,4], we need to find the electric field of a given stationary charge distribution. The purpose is to find,

$$\vec{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r'})\ (\vec{r} - \vec{r'})\ d\tau'}{|\vec{r} - \vec{r'}|^3} \tag{2.27}$$

But often it is difficult to calculate $\vec{E}$ directly using this formula. So it is best and easy to calculate the potential first and taking its negative gradient to calculate the electric field $\vec{E}$.

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r'})\ d\tau'}{|\vec{r} - \vec{r'}|} \tag{2.28}$$

In the case of problems involving conductors $\rho$ is not known. So it is useful if we re-frame the problem in differential form, using Poisson's equation,

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho \tag{2.29}$$

If $\rho = 0$, in the region where there is no charge, the Poisson's equation reduces to Laplace's equation.

$$\nabla^2 V = 0 \tag{2.30}$$

or

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Any function that has continuous partial derivatives of second order and satisfies Laplace's equation is called a Harmonic function. In terms of complex variables, if a function $f(z) = u(x, y) + iv(x, y)$ is analytic then u and v are harmonic functions.[1, 2]

### 2.2.1 Laplace's equation in one dimensions

In one dimensions Laplace's equation is simply,

$$\frac{d^2V}{dx^2} = 0$$

and evidently its solution is,

$$V(x) = mx + b \tag{2.31}$$

The solutions given by equation (2.31) have the following features:

- $V(x)$ is the average of $V(x + a)$ and $V(x - a)$ for any a.
- Laplace's equation tolerates no local maxima or minima. Extreme values hence occur at the end points. [1]

### 2.2.2 Laplace's equation in two dimensions

Here V depends on two variables and hence,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Harmonic functions in two dimensions have the following properties :

- The value of $V(x, y)$ at a point $(x, y)$ is the average of the values of V around the point $(x, y)$. That is if we consider a circle of radius R with $(x, y)$ as the centre, then the value of $V(x, y)$ is the average value of V on the circle.

$$V(r) = \frac{1}{2\pi r} \oint_{(circle)} V dl$$

- As in one dimensional case, V has no local maximum or minimum. Solution to Laplace's equation in two dimensions is the smoothest conceivable surface and it is the minimum surface spanning a given boundary line.[1]

### 2.2.3 Laplace's equation in three dimensions

In three dimensions,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Similar to one and two dimensions, the value of V at a point $\vec{r}$ is equal to the average of the values of V on a sphere of radius R with the centre at $\vec{r}$.

$$V(r) = \frac{1}{4\pi R^2} \oint_{(sphere)} V da$$

As a consequence V can have no local maxima or minima. The extreme values occur at the boundary.[1]

### 2.2.4 Boundary conditions and Uniqueness theorem

To find the complete solution to Laplace's equation we need to have boundary conditions to evaluate the constants. But how many boundary conditions are required? Is a given set of boundary conditions is sufficient to have consistent and complete solution? All these questions are answered by Uniqueness theorems. In one dimensions, it is obvious that we should have two boundary conditions such as the value of the function at the end points or the value of the function and the value of the derivative at an end point etc., But in two and three dimensions it is a difficult question to answer.

### 2.2.5 First uniqueness theorem

*The solution to Laplace's equation in some volume $\nu$ is uniquely determined if V is specified on the boundary surface S.* [1]

Corollary: The potential in a volume $\nu$ is uniquely determined if (a) charge density throughout the region and (b) the value of V on all the boundaries are specified.

### 2.2.6 Second uniqueness theorem

*In a volume $\nu$ surrounded by conductors and containing a specified charge density $\rho$, the electric field is uniquely determined if the total charge on each conductor is given.*[1]

This theorem guarantees the uniqueness of solution for electric field inside a volume V containing conducting materials having some charge. For the uniqueness we should be given with total charge present on the conductors. [1, 2]

## 2.3 General solutions of Maxwell's equations

Maxwell's equations are given by,

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho \tag{2.32}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{2.33}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{2.34}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{2.35}$$

We have to find $\vec{E}(\mathbf{r}, t)$ and $\vec{B}(\mathbf{r}, t)$ given $\rho(\mathbf{r}, t)$ and $\vec{J}(\mathbf{r}, t)$. In the static case these are determined by Coulomb's law and Biot-Savart's law. Let us generalise these to time dependent case.

In the dynamic case $\vec{\nabla} \cdot \vec{B} = 0$ still holds. But electric field has a non zero curl. Since $\vec{B}$ is still divergenceless we can write,

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Where, $\vec{A}$ is called Magnetic vector potential. With this equation (2.45) now implies,

$$\vec{\nabla} \times \left( \vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0$$

Hence we can find a scalar potential V such that

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} V$$

Therefore, finally we have,

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} V \tag{2.36}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{2.37}$$

This potential formulation of $\vec{E}$ and $\vec{B}$ readily satisfy the two homogeneous Maxwell's equations. So we shall put them in the remaining two inhomogeneous equations to get

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$$

$$\vec{\nabla} \cdot \left( -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} V \right) = \frac{1}{\epsilon_0} \rho$$

Or

$$\nabla^2 V + \frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{A}\right) = -\frac{1}{\epsilon_0}\rho \tag{2.38}$$

This equation reduces to Poisson's equation when $\vec{A}$ is a constant. In the same way equation (2.46) becomes,

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial}{\partial t}\left(-\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}V\right)$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right) = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial}{\partial t}\left(-\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}V\right)$$

Or

$$\left(\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}\right) - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}\right) = -\mu_0\vec{J} \tag{2.39}$$

Equations (2.49) and (2.50) contain all the information in Maxwell's equations.[3, **?**]

### 2.3.1 Gauge transformations

: Even though potential formulation helps us to reduce Maxwell's equations to two equations (2.49) and (2.50), still these equations are coupled second order partial differential equations which are not easy to solve. We can decouple these equations by imposing conditions on the potentials $\vec{A}$ and V by exploiting the arbitrariness involved in their definitions. The potentials as defined by equations (2.47) and (2.48) are not unique. We can find some other $\vec{A}$ and V such that they give same electric and magnetic fields. Let us denote the new potentials by $\vec{A'}$ and V'.

$$\vec{A'} = \vec{A} + \vec{\alpha} \;\; and \;\; V' = V + \beta$$

Hence,

$$\vec{\alpha} = \vec{\nabla}\lambda, \;\; where\ \lambda\ is\ a\ scalar\ fuction.$$

Since the two potentials also give the same $\vec{E}$ ,

$$\vec{E} = -\frac{\partial\vec{A'}}{\partial t} - \vec{\nabla}V'$$

$$\Rightarrow \vec{E} = -\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}V - \left\{\frac{\partial\left(\vec{\nabla}\lambda\right)}{\partial t} + \vec{\nabla}\beta\right\}$$

Hence we must have,

$$\vec{\nabla}\left\{\beta + \frac{\partial \lambda}{\partial t}\right\} = 0$$

Therefore,

$$\beta = -\frac{\partial \lambda}{\partial t} + k(t)$$

We can absorb k(t) into lambda, thus

$$\beta = -\frac{\partial \lambda}{\partial t}$$

Hence we conclude that we can find $\vec{A'}$ by adding a gradient of some scalar function $\lambda(r, t)$ to $\vec{A}$ to get the same old electric and magnetic fields provided we subtract $\frac{\partial \lambda}{\partial t}$ from V at the same time. Finally we have,

$$\vec{A'} = \vec{A} + \vec{\nabla}\lambda$$

$$V' = V - \frac{\partial \lambda}{\partial t}$$

Such transformations of potentials which give same electric and magnetic fields are called Gauge transformations. We can impose conditions on $\vec{A}$ and V using gauge transformations to simplify the problems of finding the electric and magnetic fields. [1, 3]

### 2.3.2 The Coulomb Gauge

The gauge which is chosen such that $\vec{\nabla} \cdot \vec{A} = 0$ is called the Coulomb gauge. Hence in this gauge equation (2.49) becomes,

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho$$

This is Poisson's equation which can be solved and the solution is given by,

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r'}, t)}{|\vec{r} - \vec{r'}|} d\tau'$$

The other vector potential satisfies the inhomogeneous wave equation;

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} + \frac{1}{c^2}\vec{\nabla}\left\{\frac{\partial V}{\partial t}\right\}$$

Advantage of this gauge is that the scalar potential V is easy to calculate but at the same time it is difficult to calculate the vector potential. This gauge is often used when there are no sources are present and in that case the vector potential satisfies the homogeneous wave equation. [1, 3]

### 2.3.3 The Lorenz Gauge

: In this gauge we chose $\vec{A}$ such that,

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\left\{\frac{\partial V}{\partial t}\right\} = 0$$

Hence with this gauge equation (2.50) becomes,

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} \tag{2.40}$$

Meanwhile differential equation for V equation (2.49) becomes,

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0} \tag{2.41}$$

Hence $\vec{A}$ and V are treated on equal footing by the Lorenz gauge. The same differential operator

$$\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \Box^2$$

(called the**d'Alembertian**) occurs in both the equations.

$$\Box^2 V = -\frac{1}{\epsilon_0}\rho \tag{2.42}$$

$$\Box^2 \vec{A} = -\mu_0 \vec{J} \tag{2.43}$$

The d'Alembertian is the generalization of the Laplacian and equations (2.53) and (2.54) can be regarded as four dimensional Laplace's equation. [1, 3]

# Chapter 3

# WAVE GUIDES

## 3.1 wave guides

### 3.1.1 Introduction

A wave guide is generally a linear structure that allows electromagnetic waves to pass between its ends. In most cases a wave guide is a hollow metallic pipe which is used to pass radio and microwave frequencies without much power loss. Wave guides can also be constructed by using dielectric materials depending on the frequencies to be conveyed through them. When we are working with low frequencies typically less than 200 MHz sending electromagnetic signals through either parallel transmission lines or co-axial cables is fairly common place. But once the frequency is higher we need special structures like wave guides in order to send electromagnetic waves from one place to another.

Electromagnetic wave guides are analysed by solving Maxwell's equations with boundary conditions specified by the properties of the materials used and their interfaces. These equations have many solutions which are called modes and each mode has a particular cut-off frequency below which that mode can not exists in the wave guide.

We know that electric field inside a metallic conductor is zero and hence by faraday's law the magnetic field is also zero. Then the boundary condition which come into picture are as follows.

$$\mathbf{E}^{\parallel} = 0 \tag{3.1}$$

$$H^{\perp} = 0 \tag{3.2}$$

Since metals are good conductors they have free charges on their surface and presence of a parallel electric field would set up surface currents. This leads to the first boundary condition. The absence of the parallel component of the electric field will not induce magnetic fields perpendicular to the surface of the wave guide, hence we have the second boundary condition. We assume that waves travel down the wave guide in the z-direction, the electric and magnetic fields have a time dependence of $e^{i\omega t}$ and these fields should obey Maxwell's equations which are given by,

$$\vec{\nabla} \cdot \vec{E} = 0 \tag{3.3}$$

$$\vec{\nabla} \cdot \vec{H} = 0 \tag{3.4}$$

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \tag{3.5}$$

$$\vec{\nabla} \times \vec{H} = \epsilon \frac{\partial \vec{E}}{\partial t} \tag{3.6}$$

Our task is to find $\mathbf{E}(x, y, z, t)$ and $\mathbf{H}(x, y, z, t)$ such that they obey Maxwell's equations with the boundary conditions (3.1)and (3.2).
Consider the two curl equations. Since we are assuming the time dependence of the fields as $e^{i\omega t}$, the derivative with respect to time of electric and magnetic fields is just multiplying them with $i\omega$. Hence equation (3.5) can be written as,

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} = -i\mu\omega\vec{H}$$

Taking curl on both sides we have,

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E}$$

$$= \vec{\nabla} \times \left(-\mu \frac{\partial \vec{H}}{\partial t}\right)$$

$$= -\, i\ \mu\ \omega(\vec{\nabla} \times \vec{H})$$

$$= -\, i\ \mu\ \omega(\vec{J} + i\ \epsilon\ \omega\vec{E})$$

$$= -\, i\ \mu\ \omega(\sigma\vec{E} + i\ \epsilon\ \omega\vec{E})$$

$$\nabla^2 \vec{E} = i\ \mu\ \omega(\sigma + i\ \epsilon\ \omega)\vec{E}$$

Since we are considering wave guides having air or vacuum between the plates, $\sigma$, the conductivity is essentially zero. Hence,

$$\nabla^2 \vec{E} = -\mu\epsilon\omega^2 \vec{E} \tag{3.7}$$

Similarly one can obtain the corresponding equation for the magnetic field,

$$\nabla^2 \vec{H} = -\mu\epsilon\omega^2 \vec{H} \tag{3.8}$$

In the next step we would write explicitly all the components of the equations (3.5) and (3.6) namely Faraday's law and Ampere's law respectively. We will get two sets of equations.

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = i\epsilon\omega E_x \tag{3.9}$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = i\epsilon\omega E_y \tag{3.10}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = i\epsilon\omega E_z \tag{3.11}$$

The other set of equations corresponding to Faraday's law (3.5) are given by,

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -i\mu\omega H_x \tag{3.12}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -i\mu\omega H_y \tag{3.13}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -i\mu\omega H_z \tag{3.14}$$

We will solve these sets of equations with appropriate boundary conditions to deduce the behaviour of confined electromagnetic waves. [1, 4]

### 3.1.2 Modal propagation

In free space, electromagnetic waves are transverse. But when confined they may lose this feature. In general electromagnetic waves are not transverse in wave guides and they exhibit different modes. These modes are classified as **Transverse electric mode**(TE mode), **Transverse magnetic mode**(TM mode) and **Transverse electromagnetic mode**(TEM mode). If we consider z-direction to be the direction of propagation then in TE mode the longitudinal component of the electric field $E_Z$ is equal to zero. Similarly in TM mode the longitudinal component of the magnetic field $H_z$ is zero. In very rare cases both $E_z$ and $B_z$ is zero. Such a mode is called TEM mode.

## 3.2 Parallel plate wave guide

Consider two infinite parallel perfectly conducting metal plates separated by a distance 'd' in the x-direction as shown in the figure (3.a). Assume that the plates are infinite in y and z direction and we are sending electromagnetic waves in z-direction. Since the boundaries are $x = 0$ and $x = d$, the boundary condition to be satisfied is $E_y = 0$ at $x = 0$ and $x = a$. Under these circumstances the electric field can be written in the form,

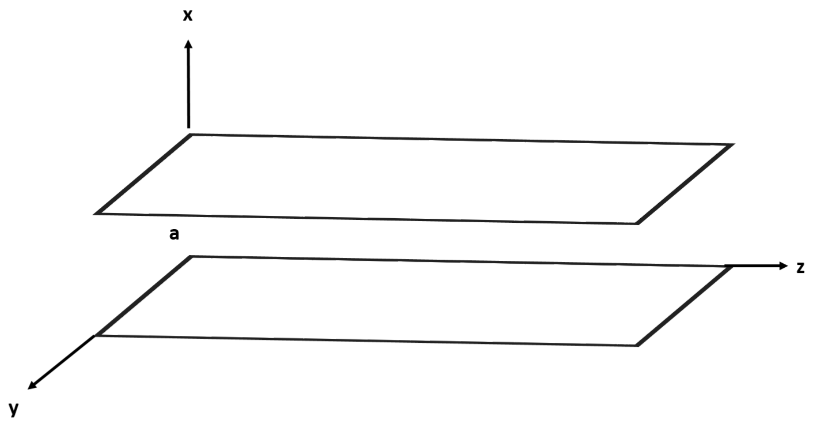


**figure (3.a): Parallel plate wave guide**

$$\vec{E}(x, y, z) = E(x, y)e^{i\omega t - \gamma z} \tag{3.15}$$

Here the exponential part stands for propagation. $\gamma$ is in fact a complex quantity given by, $\gamma = \alpha + i\beta$ where $\alpha$ stands for attenuation and $\beta$ stands for propagation. If we consider the medium between the plates to be air or vacuum this $\gamma$ would either be attenuative or propagative. There are no boundary condition to be met in y-direction and hence derivative of the fields with respect to y is zero. The form of the electric field tells us that derivative with respect to z is same as multiplying with $-\gamma$. Therefore the set of equations (3.9)-(3.11) and (3.12)-(3.14) become,

$$\gamma H_y = i\omega\epsilon E_x \tag{3.16}$$

$$-\gamma H_x - \frac{\partial H_z}{\partial x} = i\omega\epsilon E_y \tag{3.17}$$

$$\frac{\partial H_Y}{\partial x} = i\omega\epsilon E_z \tag{3.18}$$

In the same way,

$$\gamma E_y = -i\mu\omega H_x \tag{3.19}$$

$$-\gamma E_x - \frac{\partial E_z}{\partial x} = -i\mu\omega H_y \tag{3.20}$$

$$\frac{\partial E_y}{\partial x} = -i\mu\omega H_z \tag{3.21}$$

And equations (3.7) and (3.8) become,

$$\nabla^2 \vec{E} = -\omega^2 \mu\epsilon \vec{E} \tag{3.22}$$

$$\nabla^2 \vec{H} = -\omega^2 \mu\epsilon \vec{H} \tag{3.23}$$

Let us find the nature of the fields in the case of TE mode, that is when $E_z = 0$. For this case, we would write equation (3.22) for $E_y$,

$$\frac{\partial^2 E_y}{\partial y^2} + \gamma^2 E_y = -\omega^2 \mu \epsilon E_y$$

Defining $k^2 = \gamma^2 + \omega^2 \mu \epsilon$ we have,

$$\frac{\partial^2 E_y}{\partial y^2} + k^2 E_y = 0 \tag{3.24}$$

This is a well known equation whose solution is given by,

$$E_y = A \sin kx + B \cos kx$$

Applying boundary condition, $E_y = 0$ at $x = 0$ and at $x = d$ we have,

$$E_y = E_y^o \, \sin\left(\frac{n\pi x}{d}\right) \, e^{-\gamma z} \tag{3.25}$$

Where $n = 1, 2, 3......$ Obviously $n = 0$ is not possible.This equation implies that the electric field lines are dense in mid way between the plates. By using equations (3.16) to (3.21) we can find the remaining field components namely $H_x$ and $H_z$

$$H_x = \frac{-\gamma}{i\mu\omega} E_y = -E_y^o \, \frac{\gamma}{i\mu\omega} \, \sin\left(\frac{n\pi x}{d}\right) e^{-\gamma z} \tag{3.26}$$

$$H_z = \frac{-1}{i\mu\omega}\frac{\partial E_y}{\partial x} = \frac{-n\pi}{i\mu\omega d} \, E_y^o \, \cos\left(\frac{n\pi x}{d}\right) \, e^{-\gamma z} \tag{3.27}$$

Now we find out the condition for propagation of these modes. For this $\gamma$ should be purely imaginary. We defined $\gamma$ as,

$$\gamma^2 = k^2 - \mu\epsilon\omega^2$$

$$\gamma = \sqrt{k^2 - \mu\epsilon\omega^2}$$

$$\gamma = \sqrt{\left(\frac{n\pi}{d}\right)^2 - \mu\epsilon\omega^2} = i\beta$$

So for propagation,

$$\omega^2 \mu\epsilon > \left(\frac{n\pi}{d}\right)^2$$

Hence the frequency of the electromagnetic wave should be greater than some critical frequency which we call cut-off frequency for the propagation of a given mode. Otherwise

$\gamma$ will be purely real so that the mode will be attenuated. Cut-off frequency is given by,

$$\omega_c = \frac{1}{\sqrt{\mu\epsilon}}\frac{n\pi}{d} \tag{3.28}$$

Phase velocity of the wave is given by

$$\begin{aligned} v_\phi =& \frac{\omega}{\beta} \\ =& \frac{\omega}{\sqrt{\omega^2\mu\epsilon - \left(\frac{n\pi}{d}\right)^2}} \end{aligned}$$

As $\omega$ approaches $\omega_c$, phase velocity tends to infinity and for very large $\omega$, it becomes equal to the velocity of light.
We can find the components of electric and magnetic fields for TM-modes in the same way as we did for TE mode. In this case we assume that $H_z$ is zero. The resulting components are given by

$$H_y = H_y^o \cos\left(\frac{m\pi x}{d}\right) e^{-i\beta z}$$

$$E_x = H_y^o \frac{\beta}{\omega\epsilon} \cos\left(\frac{m\pi x}{d}\right) e^{-i\beta z}$$

$$E_z = H_y^o \frac{im\pi}{\omega\epsilon d} \cos\left(\frac{m\pi x}{d}\right) e^{-i\beta z}$$

As we see clearly the solutions in TM mode have cosine functions. If we put $m = 0$ the fields will not identically vanish. Hence TM mode with $m = 0$ have components,

$$H_z = 0, \quad E_z = 0, \quad H_y = H_y^o\, e^{-i\beta z}, \quad E_x = H_y^o \frac{\beta}{\omega\epsilon}\, e^{-i\beta z}$$

We can conclude TM mode corresponding to $m = 0$ is identical to transverse electromagnetic mode (TEM mode). [4]

## 3.3 Rectangular wave guide

Rectangular wave guide is a hollow metallic structure with rectangular cross section. This structure is formed by closing the ends of a parallel plate wave guide in y-direction. Let 'a' be its width and 'b' be its height as shown in the diagram (3.b).As before z is the direction of propagation. The metal plates are assumed to be perfectly conducting. The time dependence of the the fields is considered as $e^{i\omega t}$

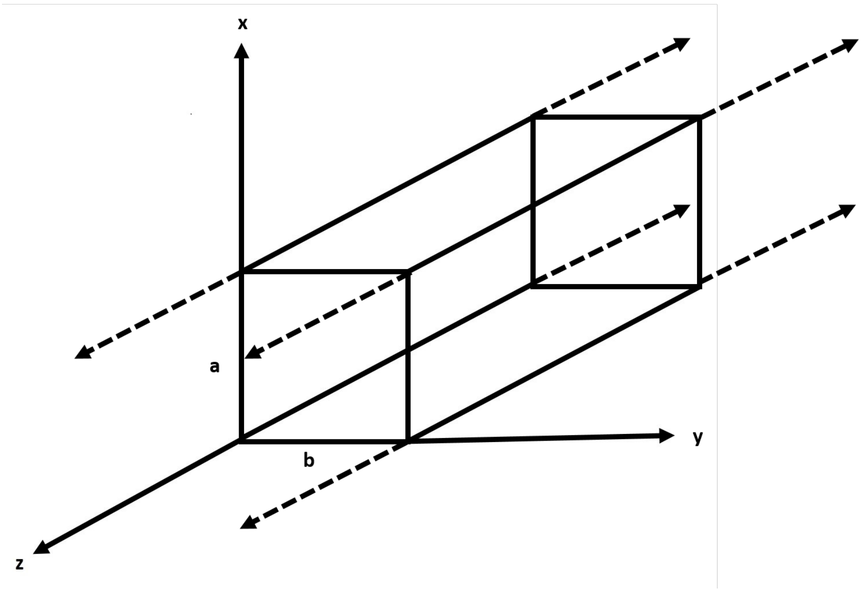


**Figure (3.b) Rectangular wave guide**

So the equation sets, (3.9)-(3.11) and (3.12-3.14) will take the following form.

$$\frac{\partial H_z}{\partial y} + \gamma H_y = i\omega\epsilon E_x \tag{3.29}$$

$$-\gamma H_x - \frac{\partial H_z}{\partial x} = i\omega\epsilon E_y \tag{3.30}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = i\omega\epsilon E_z \tag{3.31}$$

Similarly,

$$\frac{\partial E_z}{\partial y} + \frac{\partial E_y}{\partial z} = -i\omega\mu H_x \tag{3.32}$$

$$-\gamma E_x - \frac{\partial E_z}{\partial x} = -i\omega\mu H_y \tag{3.33}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -i\omega\mu H_z \tag{3.34}$$

Solving equation (3.9) and (3.13), we can express $E_x$ in terms of derivatives of $H_z$ and $E_z$. In the same way we can also express $E_y$, $H_x$ and $H_y$ in this manner. Thus we have,

$$E_x = -\frac{\gamma}{k^2}\frac{\partial E_z}{\partial x} - \frac{i\mu\omega}{k^2}\frac{\partial H_z}{\partial y} \tag{3.35}$$

$$E_y = -\frac{\gamma}{k^2}\frac{\partial E_z}{\partial y} - \frac{i\mu\omega}{k^2}\frac{\partial H_z}{\partial x} \tag{3.36}$$

$$H_x = -\frac{\gamma}{k^2}\frac{\partial H_z}{\partial x} + \frac{i\epsilon\omega}{k^2}\frac{\partial E_z}{\partial y} \tag{3.37}$$

$$H_y = -\frac{\gamma}{k^2}\frac{\partial H_z}{\partial y} - \frac{i\epsilon\omega}{k^2}\frac{\partial E_z}{\partial x} \tag{3.38}$$

Where $k^2 = \gamma^2 + \mu\epsilon\omega^2$. Now we shall find the expression for fields in TE mode. In this case $E_z = 0$. This implies $H_z \neq 0$. Therefore equation (3.2) implies,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^2\right) \tilde{H}_z(x, y) = 0$$

Here, $H_z(x, y, z) = \tilde{H}_z(x, y)\ e^{-\gamma z}$ We can solve the above equation by variable separable by taking $\tilde{H}_z(x, y) = X(x)Y(y)$. The solution will be of the form,

$$X(x) = C_1 \cos(k_x x) + C_2 \sin(k_x x)$$

$$Y(y) = C_3 \cos(k_y y) + C_4 \sin(k_y y)$$

With $k^2 = k_x^2 + k_y^2$ and hence $H_z = \{X(x)Y(y)\}$. Boundary conditions require that $E_x = 0$ at $y = 0$ and $y = b$ also $E_y = 0$ at $x = 0$ and $x = a$. For $E_x$ to be zero, $\frac{\partial H_z}{\partial y}$ must vanish at $y = 0$ and $y = b$. In the same lines,for $E_y$ to be zero, $\frac{\partial H_z}{\partial x}$ must vanish at the planes $x = 0$ and $x = a$. Applying these boundary conditions we get the following solution for $H_z$.

$$H_z = C \cos(k_x x) \cos(k_y y)\ e^{-\gamma z} \tag{3.39}$$

Here, $k_x = \frac{m\pi}{a}$ and $k_y = \frac{n\pi}{b}$. $m, n = 0, 1, 2..$ but not simultaneously zero. Once we get $H_z$ it is easy to obtain the other field components using equations (3.29) to (3.34)

$$E_x = \frac{i\omega\mu}{k^2}\frac{Cn\pi}{b} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)\ e^{-\gamma z}$$

$$E_y = -\frac{i\omega\mu}{k^2}\frac{Cm\pi}{a} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)\ e^{-\gamma z}$$

$$H_x = -\frac{\gamma}{k^2}\frac{Cm\pi}{a} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)\ e^{-\gamma z}$$

$$H_y = \frac{\gamma}{k^2}\frac{Cn\pi}{b} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)\ e^{-\gamma z}$$

Expression for cut off frequency becomes,

$$\omega_c = \frac{1}{\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \tag{3.40}$$

This is the cut-off frequency of $TE_{mn}$ mode. If we consider $TE_{10}$ mode, it has minimum cut-off frequency. Hence it is known as dominant $TE$ mode. The expressions for field components imply that $TE_{00}$ mode is not possible because all the components will identically vanish if $m = n = 0$.

It is possible to obtain the nature of electric and magnetic field components for $TM$ mode by considering $H_z = 0$ and $E_z \neq 0$. Applying necessary boundary conditions it can be shown that,

$$E_z = E_z^o \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \tag{3.41}$$

where $m = 1, 2, 3..$ and $n = 1, 2, ...$ Using this we can deduce the other field components. Since m and n can not be zero, obviously one can not have $TM_{00}$, $TM_{10}$ and $TM_{01}$ modes.$TM$ modes have a higher cut off frequency than the lowest $TE$ mode which is why $TE_{10}$ is called dominant mode.
The rectangular wave guides do not support $TEM$ modes. This is because, $TEM$ mode both $H_z$ and $E_z$ are zero which would make all other field components identically equal to zero. [4]

## 3.4 Impedance in a rectangular wave guide

The wave impedance is generally defined as the ratio of the transverse component of the electric field to the transverse component of the magnetic field. For a transverse electromagnetic wave travelling through an unbounded homogeneous medium, the wave impedance is equal to the intrinsic impedance of the medium which is given by,

$$\eta = \frac{E_x}{H_y} = \sqrt{\frac{\mu}{\epsilon}}$$

In free space the impedance of plane electromagnetic waves is given by,

$$\eta_o = \sqrt{\frac{\mu_o}{\epsilon_o}} = 376.731\ \Omega$$

### For TE mode

The expressions for $E_x$ and $H_y$ in TE mode are given by,

$$E_x = -\frac{i\omega\mu}{k^2}\frac{n\pi}{b} H_o \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-\gamma z}$$

$$H_y = -\frac{\gamma}{k^2}\frac{n\pi}{b} H_o \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-\gamma z}$$

Where

$$k^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{m\pi}{a}\right)^2$$

For propagating solution $\gamma$ should be purely imaginary,

$$\gamma = i\beta = i\sqrt{\mu\epsilon\omega^2 - k^2}$$

Therefore, the expression for impedance becomes,

$$\eta_{TE} = \frac{E_x}{H_y} = \frac{\omega\mu}{\beta} = \sqrt{\frac{\mu}{\epsilon}}\frac{1}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

For a rectangular wave guide of dimensions 2.08 $cm \times 1.01\ cm$ we have studied the variation of this impedance with frequency. The following is the MATLAB code for plotting impedance versus frequency.

```
clc;
a =0.0101;
b =.0208;
m =1;
n =0;    ( for  TE10  mode)
c =3e8;
d =1./(a. ∧ 2);
e =1./(b. ∧ 2);
fc =c. * pi. * sqrt(d + e);  (cut off frequency for TE[1,0])
f =[0.3e9 : 1e9 : 300e9];
cnst =376.731 (square root of ratio of permeability to  permittvity in vacuum)
g =(fc./f). ∧ 2;
imp =cnst./sqrt(1 − g);  (Impedance of rectangular waveguide for TE mode)
plot(f, imp);
```

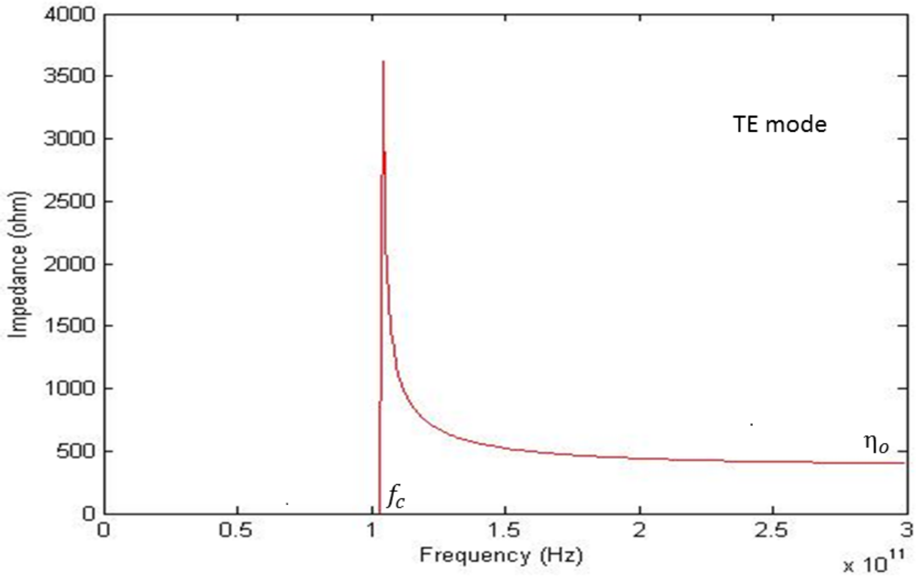


Figure (3.c): variation of impedance Vs frequency for TE mode

## For TM mode

The expressions for $E_x$ and $H_y$ in TM mode are given by,

$$E_x = -\frac{\gamma}{k^2}\frac{m\pi}{a}E_o \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-\gamma z}$$

$$H_y = -\frac{i\omega\epsilon}{k^2}\frac{m\pi}{a}E_o \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-\gamma z}$$

Where

$$k^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{m\pi}{a}\right)^2$$

As before, for propagating solution $\gamma$ should be purely imaginary,

$$\gamma = i\beta = i\sqrt{\mu\epsilon\omega^2 - k^2}$$

Therefore, the expression for impedance becomes,

$$\eta_{TM} = \frac{E_x}{H_y} = \frac{\beta}{\omega\epsilon} = \sqrt{\frac{\mu}{\epsilon}}\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

For the same rectangular wave guide of dimensions 2.08 *cm* × 1.01 *cm* we have studied the variation of this impedance with frequency. The following is the MATLAB code for plotting impedance versus frequency graph.

$clc;$

$a = 0.0101;$

$b = .0208;$

$m = 1;$

$n = 1;$ $\quad (\ for \quad TM_{11} \quad mode)$

$c = 3e8;$

$d = 1./(a. \wedge 2);$

$e = 1./(b. \wedge 2);$

$fc = c. * pi. * sqrt(d + e);$ $\quad (cut \quad off \quad frequency \quad for \quad TM[1,1])$

$f = [0.3e9 : 1e9 : 300e9];$

$cnst = 376.731$ $\quad (square \quad root \quad of \quad ratio \quad of \quad permeability \quad to \quad permittvity \quad in \quad vacuum)$

$g = (fc./f). \wedge 2;$

$imp = cnst. * sqrt(1 - g);$ $\quad (Impedance \quad of \quad rectangular \quad waveguide \quad for \quad TM \quad mode)$

$plot(f, imp);$

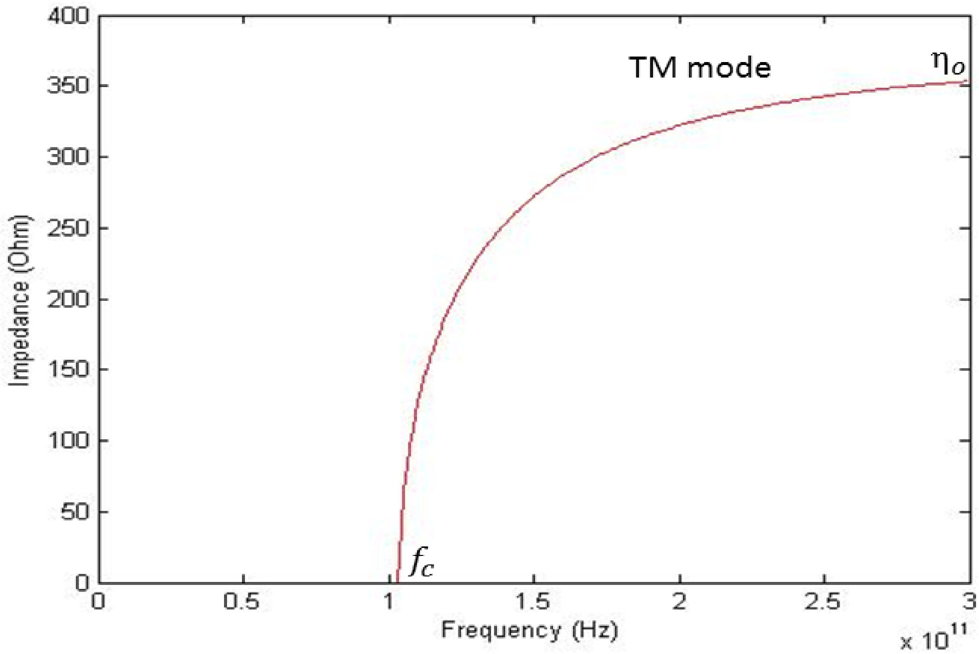


Figure (3.d): variation of impedance Vs frequency for TM mode

## Conclusions

As we see from first graph (figure 3.c),for TE mode, the impedance in rectangular wave guide decreases as the frequency is increased through cut off frequency. But for TM mode impedance increases with frequency above cut off frequency. This is shown in the second graph (figure 3.d). Comparing the two graphs, it is evident that the value of the impedance for TE mode is always higher than that for TM mode. For TE mode impedance is grater than the wave impedance in vacuum while for TM mode the impedance in the wave guide is less than that in vacuum. From the expressions for impedances, one can infer that it is imaginary below cut off frequency for both TE and TM modes. The graphs show only the variation of impedance above cut off frequency where it is no more imaginary. The real positive value of the impedance shows that wave guide is resistive and the waves carry energy.[4]

# Chapter 4

# CAVITY RESONATORS

## 4.1 Introduction

We are familiar with LCR resonant circuits which works at low frequencies. It has an inductance L, capacitance C and a resistance R which represents the losses in the circuit. The resonant frequency f is given by

$$f = \frac{1}{2\pi\sqrt{LC}}$$

If a similar resonant circuit is required at microwave frequencies, the values of L and C needed would be very small to build practically. If we try to resonate these type of circuits with high frequencies the losses represented by R will be very large. So if we need a resonant circuit at microwave frequencies we need to pay attention to cavity resonators. When a wave is propagating in a wave guide, the electric and magnetic fields exist and travel in definite patterns. If we place a metal plate across the wave guide there is complete reflection and the returning wave exhibits the same patterns. The waves travelling in opposite directions produce a standing wave and the electromagnetic wave bounces back and forth between the two plates. This is what happens in a cavity. [5]

## 4.2 Modes in cavities

Since there are an infinite number of modes in a wave guide, it follows that a cavity also has an infinite number of modes. Similar to wave guides cavities also have two types of modes, TE and TM. The modes are designated by three subscripts l,m and n. A low frequency LCR circuit has a single resonating frequency while the resonating cavities have many resonant frequencies.

## 4.3 Rectangular cavity resonator

### Mathematical analysis

Consider a rectangular cavity having dimensions a, b and d in x, y and z directions as shown in the following diagram (3.e). Here also we consider the time dependence of the fields to be $e^{i\omega t}$ and each components of electric and magnetic fields is a function of x,y and z. Let $E_\alpha$ be any component of the electric field inside the cavity. Then it should satisfy,

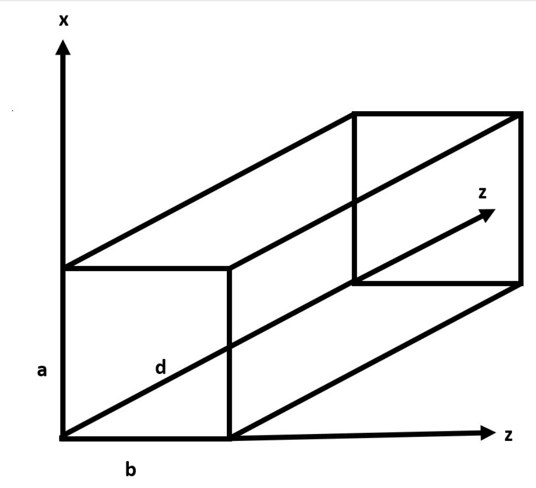


**Figure (3.e) Rectangular cavity**

$$\nabla^2 E_\alpha = -\omega^2 \mu\epsilon E_\alpha \tag{4.1}$$

Applying method of separation of variables,

$$E_\alpha = X_\alpha(x) Y_\alpha(y) Z_\alpha(z)$$

Substituting this in the above differential equation we get,

$$\frac{1}{X_\alpha}\frac{\partial^2 X_\alpha}{\partial x^2} + \frac{1}{Y_\alpha}\frac{\partial^2 Y_\alpha}{\partial y^2} + \frac{1}{Z_\alpha}\frac{\partial^2 Z_\alpha}{\partial z^2} = -\omega^2\mu\epsilon$$

Each one of the terms in left hand side should be equal to a constant.

$$\frac{1}{X_\alpha}\frac{\partial^2 X_\alpha}{\partial x^2} = -k_x^2 \quad \frac{1}{Y_\alpha}\frac{\partial^2 Y_\alpha}{\partial y^2} = -k_y^2 \quad \frac{1}{Z_\alpha}\frac{\partial^2 Z_\alpha}{\partial z^2} = -k_z^2 \tag{4.2}$$

$k_x$, $k_y$ and $k_z$ are related by the following equation.

$$k_x^2 + k_y^2 + k_z^2 = \omega^2 \mu \epsilon \tag{4.3}$$

Solving the equations (4.2) we get

$$E_\alpha(x, y, z) = (A_\alpha \cos(k_x x) + B_\alpha \sin(k_x x)) (C_\alpha \cos(k_y y) + D_\alpha \sin(k_y y)) \\ (F_\alpha \cos(k_z z) + G_\alpha \sin(k_z z))$$

In particular,

$$E_x(x, y, z) = (A_x \cos(k_x x) + B_x \sin(k_x x)) (C_x \cos(k_y y) + D_x \sin(k_y y)) \\ (F_x \cos(k_z z) + G_x \sin(k_z z))$$

$E_x$ should become zero at $z = 0$ and $z = d$, also at $y = 0$ and $y = b$. Applying these boundary conditions we end up with,

$$E_x(x, y, z) = [(A_x \cos(k_x x) + B_x \sin(k_x x)] \ \ \sin(k_y y) \ \ \sin(k_z z) \tag{4.4}$$

With $k_z = \frac{n\pi}{d}$ and $k_y = \frac{m\pi}{b}$ where $m, n = \pm 1, \pm 2...$
Similarly we can work out for $E_y$ and $E_z$,

$$E_y(x, y, z) = [(C_y \cos(k_y y) + D_y \sin(k_y y)] \ \ \sin(k_x x) \ \ \sin(k_z z) \tag{4.5}$$

$$E_z(x, y, z) = [(F_z \cos(k_z z) + G_z \sin(k_z z)] \ \ \sin(k_x x) \ \ \sin(k_y y) \tag{4.6}$$

Where $k_x = \frac{l\pi}{a}$ with $l = \pm 1, \pm 2...$

Irrespective of the solution we got, fields must satisfy Maxwell's equations inside the cavity. In particular, the electric field must satisfy the condition $\vec{\nabla} \cdot \vec{E} = 0$ at every points inside the cavity. That is,

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$$

Thus,

$$[-A_x \cos(k_x x) + B_x \sin(k_x x)] \ \ \sin(k_y y) \ \ \sin(k_z z) \\ + [C_y \cos(k_y y) + D_y \sin(k_y y)] \ \ \sin(k_x x) \ \ \sin(k_z z) \\ + [(F_z \cos(k_z z) + G_z \sin(k_z z)] \ \ \sin(k_x x) \ \ \sin(k_y y) = 0$$

Therefore,

$$\text{At } (0,y,z)\ ,\ \vec{\nabla}\cdot\vec{E}=0 \ \text{ implies } \ B_x=0$$

$$\text{At } (x,0,z)\ ,\ \vec{\nabla}\cdot\vec{E}=0 \ \text{ implies } \ D_y=0$$

$$\text{At } (x,y,0)\ ,\ \vec{\nabla}\cdot\vec{E}=0 \ \text{ implies } \ F_z=0$$

Hence the expression for electric field inside the rectangular cavity becomes,

$$\vec{E} = E_x^o\cos(k_x x)\sin(k_y y)\sin(k_z z)\hat{x} + E_y^o\cos(k_y y)\sin(k_x x)\sin(k_z z)\hat{y} + E_z^o\cos(k_z z)\sin(k_x x)\sin(k_y y)$$

Where $k_x = \frac{l\pi}{a}$, $k_y = \frac{m\pi}{b}$, $k_z = \frac{n\pi}{d}$. l, m and n are integers which can not be zero simultaneously.

If we take $E_z = 0$ then it is called a $TE_{lmn}$ mode. So for a $TE_{lmn}$ mode the electric field components are,

$$E_x = E_x^o\cos(k_x x)\sin(k_y y)\sin(k_z z) \tag{4.7}$$

$$E_y = E_y^o\cos(k_y y)\sin(k_x x)\sin(k_z z) \tag{4.8}$$

The magnetic field components are obtained by using Faraday's law.

$$H_X = \frac{1}{i\mu\omega}E_y^o\sin(k_x x)\cos(k_y y)\cos(k_z z) \tag{4.9}$$

$$H_y = -\frac{1}{i\omega\mu}E_x^o k_z\cos(k_x x)\sin(k_y y)\cos(k_z z) \tag{4.10}$$

$$H_z = -\frac{1}{i\omega\mu}\left[E_y^o k_x - E_x^o k_y\right]\cos(k_x x)\cos(k_y y)\sin(k_z z) \tag{4.11}$$

From equation (4.3) we have,

$$k_x^2 + k_y^2 + k_z^2 = \omega^2\mu\epsilon$$

$$\frac{l^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2} + \frac{n^2\pi^2}{d^2} = \omega^2\mu\epsilon$$

Thus frequencies that are permitted are very specified frequencies which are decided by the value of m,n and l. In the case of rectangular wave guide we had seen that there was a minimum frequency above which propagation takes place. But in this case there is a specified frequency at which the propagation takes place for a given mode.The modes are called resonant modes and hence the name resonant cavity. [4, 5]

## 4.4 Optical cavities

An optical cavity or optical resonator is an arrangement of mirrors that forms a standing wave cavity resonator for light waves. Light confined in the cavity reflect multiple times producing standing waves for certain resonance frequencies. only certain patterns and frequencies of radiation will be sustained by the resonator, with the others being suppressed by destructive interference. In general, radiation patterns which are reproduced on every round-trip of the light through the resonator are the most stable, and these are the eigen modes of the resonator. Resonator modes can be divided into two types: longitudinal modes, which differ in frequency from each other; and transverse modes, which may differ in both frequency and the intensity pattern of the light.[6]

### 4.4.1 Optical Microcavities

An optical microcavity is a structure formed by reflecting faces on the two sides of a spacer layer or optical medium. The name microcavity stems from the fact that it is often only a few micrometers thick, the spacer layer sometimes even in the nanometer range. As with common lasers this forms an optical cavity or optical resonator, allowing a standing wave to form inside the spacer layer.Like its acoustic analogue the tuning fork, the optical microcavity (or microresonator) has a size-dependent resonant frequency spectrum. Microscale volume ensures that resonant frequencies are more sparsely distributed throughout this spectrum than they are in a corresponding macroscale resonator. An ideal cavity would confine light indefinitely (without loss) and would have resonant frequencies at precise values. Deviation from this ideal condition is described by the cavity Q factor (which is proportional to the confinement time in units of the optical period).[7]

### 4.4.2 Spontaneous emission in optical resonators

In its standard description, spontaneous emission is the irreversible emission of a photon into the free space modes of the electromagnetic field, accompanied by a transition of the atom from an electronic state of energy $E_2$ to one of lower energy $E_1$. The frequency of the emitted light is $\frac{E_2-E_1}{\hbar}$ where $\hbar$ is the Planck's constant.

The presence of Planck's constant in this frequency clearly indicates that the spontaneous emission is an intrinsically quantum mechanical process. Indeed, its proper description the quantisation of both the atoms and the field. A well known result of this theory is that rate of spontaneous emission is proportional to the free space mode density of the electromagnetic field.

But this description of spontaneous emission is not general and that spontaneous emission is not an intrinsic atomic property : rather, it can be modified by tailoring the electromagnetic environment that the atom can radiate into, This was first realized by Purcell, who noted that the spontaneous emission rate can be enhanced for an atom placed inside a cavity with one of its modes resonant with the transition under consideration, and by Kleppnar, who discussed the opposite case of inhibited spontaneous emission. It has also been recognised that spontaneous emission need not be an irreversible process. An atom coupled to a single mode electromagnetic field undergoes a periodic exchange of excitation between the atom and the filed.[8]

## Future work

We shall investigate spontaneous emission in micro cavities and vacuum Rabi oscillations in a single two level atom coupled to a micro cavity.

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